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CIV. *Convective Cooling of Wires in Streams of Viscous Liquids.* By A. H. DAVIS, M.Sc.*

[From the National Physical Laboratory.]

Summary.

STUDY has been made of the cooling of wires immersed in streams of various liquids. *Theoretical considerations* point to a formula of the type

$$H/k = F(vdc/k; k/cv),$$

where H = Heat loss per unit time per unit length of the wire per degree temperature elevation,
 k = Thermal conductivity of the fluid,
 c = Specific heat of the fluid per unit volume,
 ν = Kinematical viscosity of the fluid,
 v = Velocity of the fluid stream,
 d = Diameter of the wire.

Since (k/cv) is practically constant for gases, the present experiments with liquids were undertaken to reveal its importance.

The *apparatus* consisted of a motor-driven whirling arm which carried the test-wire and forced it continually through the liquid contained in an annular circular trough. Observation was made of the temperature of the wire and of the electric energy necessary to maintain a measured temperature difference between the wire and the general body of the liquid. Wires of three sizes were used, and various *liquids* whose physical constants were measured.

The *results* show that the above formula is fairly satisfactory,

* Communicated by the Director of the National Physical Laboratory.

except that a factor must be included to cover certain small residual effects dependent upon d (and possibly upon θ) and due to some cause, possibly free convection, not fully investigated. It was found that the function F could not be represented by the product of two independent terms in $vd\epsilon/k$ and k/cv respectively, but that each influences to some extent the effect of the other. The following form was found to be satisfactory:—

$$H/k = R \cdot f(k/cv) \cdot F\{(vd\epsilon/k) \times (k/cv)^{\frac{1}{2}}\},$$

where R represents the residual factors referred to. Curves are given showing the forms of the two functions. If they are represented by power terms thus, $f(x)=x^{-m}$ and $F(x)=x^n$, m and n vary with circumstances but lie within the limits given by $0 < m < 0.2$ and $0.38 < n < 0.48$.

Concise tables present the experimental results. An estimate was made of the *velocity of the natural convection current* set up by the heated wire.

CONTENTS.

INTRODUCTION.

I. THEORETICAL FORMULÆ.

II. EXPERIMENTAL.

1. Introduction.

2. Choice of Method.

3. Apparatus.

- (i.) The Whirling Arm and the Trough.
- (ii.) The Wire Holder.
- (iii.) Electrical Measurements.
- (iv.) The Resistance Bridge.
- (v.) Precautions and adjustments.
- (vi.) Procedure during an experiment.
- (vii.) Measurement of Velocity.
- (viii.) Velocity of the Natural Convection Current from the Wire.

4. The Liquids used.

- (i.) Their nature and condition.
- (ii.) Physical Properties of Liquids used.
 - (a) Density (ρ).
 - (b) Viscosity (η) and Kinematical Viscosity ($\nu = \eta/\rho$).
 - (c) Specific Heat (s and c).
 - (d) Thermal Conductivity (k).
 - (e) The ratio (k/cv).

5. The Wires used.

- (i.) Calculation of the heat-loss from the Wire.
- (ii.) Calculation of the temperature excess of the Wire.
- (iii.) Correction due to radial temperature gradient in the Wire.
- (iv.) Correction for conduction by the potential leads and terminals.

6. Results.

- (i.) Tables and Curves showing heat losses obtained with each of the liquids.
- (ii.) Empirical relations.
- (iii.) Formula covering all results :

$$H/k = A \cdot B \cdot C \cdot F \{ vdc/k \} \times (k/cv)^{\frac{1}{2}}.$$

- (a) The dependence of A upon temperature variations in the physical properties of the liquid.
- (b) Variation of B with wire diameter.
- (c) Relation between C and the nature of the liquid.
- (d) Curve showing correlation effected.

INTRODUCTION.

THE subject of the cooling of the surface of a hot body in contact with a moving fluid has received considerable investigation during recent years. For various reasons, diverse experimental methods have been used. Measurements have been made on the one hand of the heat which passes from the walls of a hot tube to the fluid flowing within it, and on the other of the cooling of wires and cylinders when immersed transversely in a fluid stream. In neither case does published information appear sufficient for determining the effect of the nature of cooling liquids: indeed, for the wire method, while air-cooling has been extensively studied, few other data are available at all.

Until liquids have received adequate attention, the laws of convection cannot be fully revealed. For convection of heat is ultimately a molecular phenomenon, which presents complexities in the case of liquids which are almost entirely absent when the gaseous state is considered.

A study has therefore been made of the cooling of wires when immersed in streams of various liquids of known physical properties.

I. THEORETICAL FORMULÆ.

Osborne Reynolds*, from the similarity of molecular motions involved in conduction of heat and diffusion of momentum (skin friction) indicated a relation between surface cooling and skin friction. His theory has received much attention in connexion with the turbulent flow in pipes. As Reynolds himself pointed out, its simplest form

* Reynolds, Proc. Manchester Lit. & Phil. Soc. (1874). See also Stanton, Report Adv. Comm. Aeronautics, p. 45 (1912-13).

requires supplementing by an additional factor relating to the transfer of heat through the eddy-free layer near the surface of the tube. G. I. Taylor * has recently calculated this correction from an estimate of the thickness of the film. He made use of the Lorentz criterion for steady fluid motion between two parallel planes moving tangentially to each other. His correction depends upon the ratio of the rates of diffusion of heat and of momentum in the fluid †.

Boussinesq ‡ was the first to give a formal theory of convection, and his analysis combined hydrodynamical equations with the Fourier equation for heat-flow. With certain assumptions he obtained, for similar bodies similarly immersed in infinite inviscid streams, a general solution which may be written in the form :

$$hl/k\theta = F(vlc/k), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where h = heat loss per second per unit area of the hot body,

l = linear dimensions of the body,

θ = temperature difference between the surface of the hot body and the distant fluid,

v = velocity of the fluid stream,

k = conductivity of the fluid,

c = capacity for heat of the fluid per unit volume.

For horizontal wires of diameter " d ," transverse to the fluid stream, and of a length sufficient to render the effects of the ends negligible, the above equation may be written

$$H/k = F(vdc/k), \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

where H = heat loss per unit time per unit length of the wire per degree temperature excess.

For the case of such thin wires, and for non-turbulent flow, Boussinesq showed the form of the function to be given by a square-root law §. King ||, following him, calculated the form

$$H/k = 1 + (2\pi)^{\frac{1}{2}}(vdc/k)^{\frac{1}{2}} = 1 + 2.51(vdc/k)^{\frac{1}{2}} \quad . \quad . \quad (2)$$

for velocities above a (small) critical value. He assumed constant flux of heat across the surface of the wire.

* Taylor, Report Adv. Comm. Aeronautics, ix. p. 423 (1916-17).

† That is, upon the variable k/cv referred to later in this paper.

‡ Boussinesq, *Comptes Rendus*, cxxii. p. 1382 (1901); cxxxiii. p. 257 (1901).

§ Boussinesq, 'Théorie analytique de la Chaleur,' t. ii. (1903). See also Russell, Phil. Mag. xx. p. 591 (1910), where exhaustive references are given.

|| King, Phil. Trans. A. ccxiv. p. 373 (1914).

Rayleigh*, in illustration of the use of the Principle of Similitude, derived an extension of the general formula, to include viscous fluids. The result may be written†:

$$hl/k\theta = F(vlc/k; k/cv) \quad \text{or} \quad H/k = F(vdc/k; k/cv). \quad (3)$$

Nusselt‡ in 1909 had already applied the principle to the problem of heat transfer in tubes, and had obtained a result of the same form. By a further assumption, he reduced his result to Boussinesq's form (equation 1).

Equation 3 forms the basis of the present work, where the effect of the k/cv term has been studied. An earlier paper§ showed how the expression follows from Boussinesq's method of analysis when a viscosity term is introduced.

II. EXPERIMENTAL.

1. INTRODUCTION.

Many experiments have been carried out on the heat-transmission to fluids flowing through hot pipes. Comparison with theory is not straightforward, since a certain length of leading-in pipe must be traversed before velocity conditions attain their final distribution||. Leprince-Ringuet¶ made a study of experiments by many observers (Nusselt, Jordan, Stanton, Carcanagues, Ser, Pennsylvania Rrd, and Henry) and obtained a formula covering all results for air. However, no formula of Nusselt's simple type would include known values for other fluids, such as certain common gases, steam, and water. Stanton** found Reynold's formula to be satisfactory in certain respects, but to need correction presumably for the heat transfer through the eddy-free surface layer. The correction term evaluated by G. I. Taylor is a function

* Rayleigh, *Nature*, xcv. p. 66 (1915).

† This formula may equally well be written $hl/k\theta = F\{vl/\nu; k/cv\}$. In this form the relation to fluid-resistance formulæ is emphasized. The mechanical resistance R per unit area is given by $R/\rho v^2 = f(vl/\nu)$. Thus heat loss and mechanical resistance are closely related, provided allowance is made for complexity of fluid structure by means of a k/cv term.

‡ Nusselt, *Zeitschrift des Vereines deutsche Ingenieure*, Oct. 23, 1909.

§ Davis, *Phil. Mag.* xlv. p. 940 (1922).

|| See Stanton, *Report Adv. Comm. Aeronautics*, viii. p. 16 (1916-17).

¶ Leprince-Ringuet, *Comptes Rendus*, clii. p. 436 (1911).

** Stanton, *Report Adv. Comm. Aero*, p. 45, 1912-13. See also *Phil. Trans. A.* exc. p. 67 (1897).

of k/cv and appears promising*, but apparently more experimental data are required †.

McAdams and Frost ‡, in a survey of certain experiments on heat transfer in pipes, give a curve covering water and certain oils, in which the heat-loss ($hl/k\theta$) is a function of the hydrodynamical variable (vl/ν). Though the range of viscosity covered appears to be small (about 3-fold), this indicates that any other terms in the complete equation are of rather subsidiary importance.

Generally speaking, therefore, experiments with tubes appear to indicate that the complete formula for heat loss contains a k/cv term. The omission of this term yields simpler formulæ, which give results which are generally of the right order but which may differ quite appreciably from measured values.

As regards the heat losses from wires and cylinders in a stream of fluid, King § has made a very comprehensive study of the cooling of fine wires in air, covering a wide range of conditions. He found his results agreed excellently with a formula very similar to his own theoretical equation. A small correction term was necessary, depending upon the temperature excess θ of the wire; this he supposed to be due to variations in the physical properties of air as the temperature is changed. He found also that the dependence upon the diameter of the wire was rather *greater* than the theoretical formula suggested. His paper contains reference to similar work by other experimenters.

Hughes || experimented with large steam-heated cylinders up to 5 cm. diameter. His data do not fit King's special formula, as the variation with velocity ($v^{0.55}$ to $v^{0.7}$) is in excess of that given by a square-root law. However, the more general form of equation 1 appears satisfactory, except that variation with diameter in this case tends to be less than the formula requires. It is noteworthy that this deviation from the formula is in the opposite sense to that found by King for his wires.

The range of published data is further extended by the results of Kennelly and Sanborn ¶, who used air at various pressures. Some earlier results by Kennelly** cover some

* Taylor, *loc. cit.*

† Stanton, Report Adv. Comm. Aero. p. 16, 1916-17. See also Pannell, p. 22.

‡ McAdams & Frost, Journ. Ind. Eng. Chem. xiv. p. 1101 (1922).

§ King, *loc. cit.*

|| Hughes, Phil. Mag. xxxi. p. 118 (1916).

¶ Kennelly & Sanborn, Am. Phil. Soc. Proc. liii. p. 55 (1914).

** Kennelly, Trans. A.I.E.E. xxviii. (1) p. 363 (1909).

of the range later studied by King, but include additional data for a larger wire.

The present author* has given a graph upon which results of these experimenters have been plotted, and it appears that equation 1*a* is fairly satisfactory, viz. :

$$H/k = F (vdc/k) \dots \dots \dots (4)$$

However, since k/cv is practically a constant for gases, such experiments cannot reveal the effect of this term. Practically no values are available for liquids, and the need of the present investigation is thus clear.

2. CHOICE OF METHOD.

The following general considerations led to the choice of the present method of investigating the cooling powers of streams of liquids. By using wires as the hot bodies size was easily altered, and by adopting electrical heating measurements of temperature and heat loss were readily made. By moving the wire through the liquid instead of the liquid past the wire, complications were avoided due to non-uniform flow in tubes and troughs, for in these the flow is greatest in the centre and is not steady until considerable lengths have been traversed. An annular circular track was used ; it has various advantages. Since some general circulation of the liquid was of course set up, the velocity distribution question was not entirely eliminated, but was only of secondary importance, as it only affected a correction term which had to be applied to the observed speed of the wire.

3. APPARATUS.

The apparatus used consisted of a motor-driven whirling arm which, by suitable attachments, carried the test wire in a vertical position and forced it continually through the liquid contained in the annular circular trough. Observation was made of the velocity of the wire and of the electric energy necessary to maintain a measured temperature difference between the wire and the general body of the liquid. The energy dissipation was determined from the observed current through the wire, and from its resistance as obtained by means of a Kelvin double bridge. This resistance gave also the temperature of the wire.

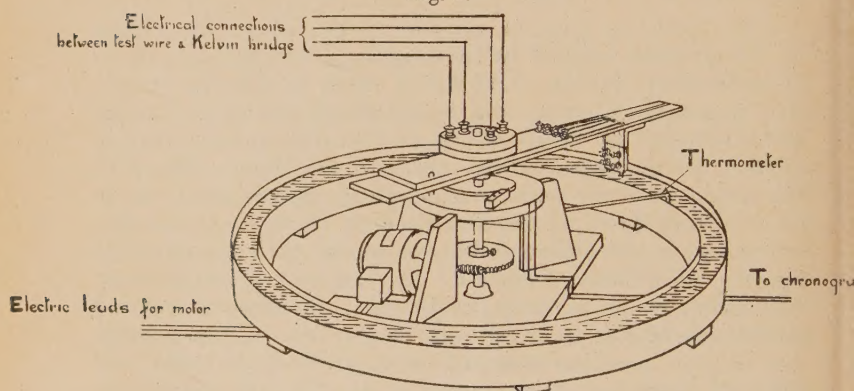
* Davis, *Phil. Mag.* xli. p. 899 (1921).

(i.) *The Whirling Arm and the Trough.*

The general arrangement of the apparatus is shown in fig. 1.

A balanced whirling arm was driven through a worm and wheel (ratio 100:1) by means of a 1/10 horse-power electric motor, fitted with ball bearings. The motor was shunt-wound, and the speed was controlled by means of a series resistance in the armature circuit. To avoid communication of vibration to the rotating arm, the motor was separately screwed down to the concrete floor. Considerable care was taken to ensure uniformity of motion, even at low speeds, throughout a complete revolution of

Fig. 1.



The Whirling Arm and the Trough.

the whirling arm. The motor was driven by means of a storage battery. The rotating arm was fitted with a friction brake to reduce to a minimum the effect of any frictional variations in the bearings. (A thrust-ball bearing took the weight of the rotating arm.) The shaft was accurately vertical, so that the rotating arm had no lowest position at which it would tend to stop during a revolution even if the balance were not exact. A spirit-level, used on the arm at various positions of a revolution, was useful in this connexion. All woodwork was painted or varnished to avoid defective alignment arising from warping.

The annular circular trough was made from sheet brass (0.16 cm. thick), and was supported separately from the concrete floor to avoid vibrations and ripples. Porcelain supports afforded thorough electrical insulation. The cross-section of the trough was 7 cm. wide and 11 cm. deep.

The test wire moved centrally in the trough, the circumference of the circular path being 242 cm. No serious difficulty was experienced on account of any variation in the width of the trough at different parts of the circumference. Some slight imperfection in this respect merely caused the galvanometer-spot to wander slightly from side to side during a revolution of the arm, instead of remaining exactly at zero all the time. Measurements of the general swirl of liquid in the centre of the trough were made at various points of the circumference, but no serious divergences from the mean were observed.

During the motion of the rotating arm the surface of the liquid was disturbed by the passage of the wire, but the effect was soon damped out and the surface was smooth again before the arm re-passed the same point on its next revolution. The wave behind the arm was especially pronounced for water, but even then, except at highest speeds (about 55 cm. per sec.), the disturbance did not persist for the time taken by the arm in a complete revolution.

Undoubtedly the use of an annular trough rather than a simple circular bath tended to reduce the amount of swirl set up as well as the persistence of these surface-disturbances.

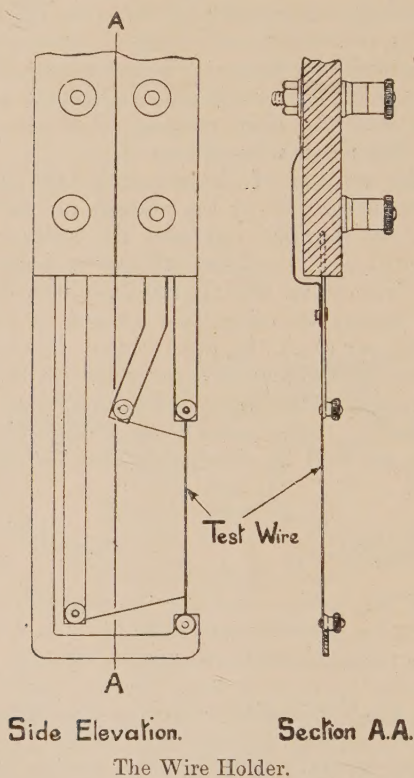
(ii.) *The Wire Holder.*

Details of the holder used for the test wire are shown in fig. 2. It was made from brass strip ($\frac{1}{4}$ in. \times $\frac{1}{16}$ in.), together with suitable terminals and an ebonite support. The holder was sufficiently springy to keep the wire gently taut, when adjusted to proper length between the terminals. Provision was made for potential and current leads. The strip moved edge-first through the liquid, and the section thus presented was found to be sufficiently stream-line in shape to be satisfactory for the velocities used.

It will be seen from the figure that the two long supporting strips were placed an inch or so behind the test wire. It appears that at the side or in front they would have been more likely to interfere with the stream past the wire. Visual observation of the motion through liquids confirmed the opinion that, upstream, the effects of the strip were limited to only a few millimetres. The arrangement adopted had the additional advantage that the whole of the immersed parts were moving centrally in the trough,

so that the velocity distribution in the fluid was simpler, the trough could be made narrower, and economy was thus effected in the quantity of liquid used.

Fig. 2.



(iii.) *Electrical Measurements.*

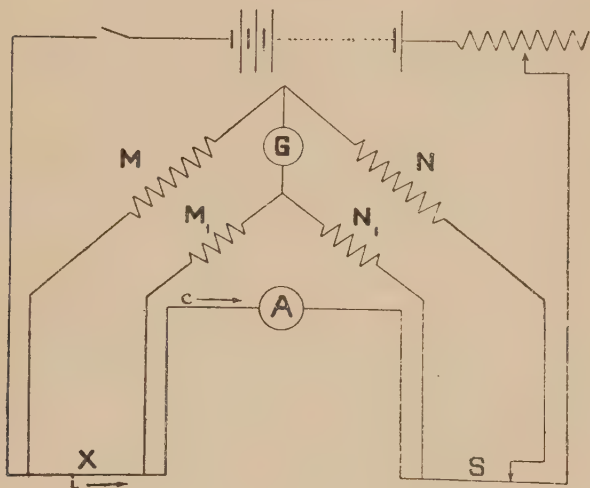
With regard to the electrical measurements, a dual arrangement was fitted up so that two methods could be used. In one a specially-constructed Kelvin bridge was used, and for the other a calibrated potentiometer was available with suitable standard resistances. On account of its greater convenience, the Kelvin-bridge method was almost invariably adopted for each series of observations; but the potentiometer was permanently installed so as to be readily available for checking the bridge, the resistances of its slide-wire and coils, and the accuracy of the ammeter.

However, for paraffin and for air, a series of potentiometer observations of heat-loss were made, and they agreed satisfactorily with those obtained by means of the Kelvin bridge.

(iv.) *The Resistance Bridge.*

Fig. 3 is a diagram of the connexions of the Kelvin bridge. The arm X represents the platinum wire under test, which was compared with the variable standard resistance S. This standard consisted of a number of 0.05-ohm

Fig. 3.



The Kelvin Bridge Connexions.

coils of heavy manganin wire (S.W.G. 11), together with a 0.1-ohm slide wire 160 cm. long. It had been carefully calibrated by a potentiometer method to read to one ten-thousandth part of an ohm. The ratio arms M, N, M_1 , and N_1 contained accurately adjusted coils of 100 ohms each, together with leads which, although they were of small resistance ($\frac{1}{2}$ ohm), had also been adjusted to sufficient equality (1 per cent.) to avoid appreciable errors which otherwise might have been introduced.

To provide electrical connexions to the wire X on the rotating arm, slip rings were utilized, similar to those described by King*, consisting of stationary copper rings

* King, *loc. cit.*

dipping into rotating mercury troughs. The resistance introduced into each arm by the slip-ring apparatus was less than 0.003 ohm, and it remained constant to less than 0.0001 ohm whether the arm was rotating or still.

The ammeter A was a Weston Laboratory Standard instrument. The galvanometer G was a reflecting instrument of the moving-coil type, very free from troublesome zero-drift. Its sensitivity was such that 1 microampere gave a deflexion of 50 millimetres on a scale at a distance of 1 metre from the galvanometer. The full sensitivity was required only when the initial resistance of the cold wire was determined, and it was suitably reduced when the heating currents were used.

When the bridge was balanced the various resistances satisfied the following formula :

$$X = \frac{M}{N} S + \frac{AN_1}{A + M_1 + N_1} \left(\frac{M}{N} - \frac{M_1}{N_1} \right). \quad . \quad . \quad (5)$$

The current c measured by the ammeter was not exactly the same as that i through the test wire. Strictly a correction was necessary according to the equation

$$i = c \left(1 + \frac{A}{M_1 + N_1} \right); \quad . \quad . \quad . \quad (6)$$

but A was usually so small that the correction was negligible.

With the bridge out of balance by an amount dX in the test wire, the galvanometer current (g) was given approximately by the formula

$$g \left\{ 2 + G \left(\frac{1}{M} + \frac{1}{N} \right) \right\} = \frac{cdX}{M}. \quad . \quad . \quad . \quad (7)$$

(v.) *Precautions and adjustments.*

At the outset the apparatus was well insulated. It was tested by means of an Evershed "Megger" Insulation tester, and also by observing whether the galvanometer remained at its true zero when only one pole of the battery was connected to the apparatus.

Thermoelectric effects were practically negligible. The connexions of the platinum test wire were all immersed in the liquid in the trough, and were thus at constant temperature. The rest of the bridge connexions were of copper or manganin, and these metals together exhibit little thermoelectric activity. Some slight effects were found on one or two occasions, but they were only appreciable when getting the original balance with a small current: they were eliminated by the familiar

device of working with the galvanometer circuit always closed, and adjusting the bridge so that no deflexion occurred when the battery was reversed. Reversals with the large currents showed no change of balance conditions.

To keep the current as steady as possible the source of supply was a 110-volt storage battery in series with a ballast resistance.

The complete ratio arms M and N, consisting of coils and leads, were accurately adjusted to equality in the following manner. The coils themselves were tested for equality by reversal of connexions, and found to be satisfactory. The potential leads from the coils, which are also in the ratio arms, were brought to equality by adjusting so that the balance of the bridge was not appreciably disturbed if the coils were short-circuited by plug contacts, thus leaving the leads themselves as the complete outer ratios of the bridge. The inner ratio arms were then similarly treated. This procedure ensured that the ratios M/N and M_1/N_1 were accurately equal to unity, and so resulted in the simple relation $X=S$, independent of the intermediate resistance A. It was experimentally verified that increasing A fivefold (from 0.2 ohm to 1.2 ohm) did not affect the reading by more than one part in 3000 (see equation 5). It was easy to make adjustment of this accuracy whenever the test wire was changed, and the apparatus was frequently checked in this and in other ways during use.

As an overall test of the bridge, a standard 0.1 ohm coil was substituted for the test wire. The Kelvin bridge indicated the correct resistance for this, and the reading was the same for large currents (5 amps.) and for very small ones (0.07 amp.); was independent of whether the arm was rotating or not; was not influenced by the speed; and was also independent of the direction of the current employed. This is a stringent test of the suitability of the bridge and slip rings for the currents involved, and of freedom from errors due to leakage or thermoelectricity.

It remained only to substitute the actual test wire and holder for the standard 0.1 ohm coil, and to test for the absence of thermal effects and leaks by observing the behaviour of the galvanometer spot on reversal of the various currents and upon switching off completely.

(vi.) *Procedure during an experiment.*

To obtain a series of observations the usual procedure was as follows. With the test wire in position, and with a small current through the wire (0.07 amp.), the bridge was

balanced by adjustment of the position of the slider along the slide wire. As the heating effect of such a small current was negligible*, this reading gave the resistance of the wire at the temperature of the oil.

The slider position was then altered to increase the reading by a definite amount dR . This disturbed the balance of the bridge. With the arm rotating at measured speed, note was made of the increased current through the wire necessary to restore the balance. From the constants of the test wire the increment of temperature associated with the resistance dR was known, and the heat loss from the wire at this speed was readily calculated.

This procedure was repeated with the same value of dR , but with several different speeds. Further series were then taken, until a range of values of temperature excess and velocity had been obtained.

During a series of experiments, the initial temperature of the liquid sometimes drifted slightly. This was allowed for by slight change in the initial slider position, so that dR was maintained constant throughout, and the drawing of a series of curves was facilitated.

It was important to remove stray suspended fluff from the liquid under test, for otherwise readings could not be taken owing to a pronounced drift of the galvanometer reading. It was astonishing how much fine fibrous material could gain entrance to the liquid, and equally remarkable that the galvanometer spot could indicate so readily when a single fibre, almost invisible, became attached to the moving wire. Consequently all liquids used were passed through filter-paper two or three times, and collected in large clean glass vessels. Before each filling, the experimental trough was very carefully cleaned with suitable agents and swilled out with petrol. A blowpipe-flame was then passed rapidly over the interior to burn off any adherent fibres, and the trough was again washed out, first with filtered petrol and finally with a small quantity of the liquid to be tested. Except when experiments were actually in progress the trough was kept covered, and at all times unnecessary disturbance of the air of the room was avoided.

Even with these precautions fluff was not completely eliminated, but the trouble was not serious. Consistent results were obtained by cleaning the test wire gently with another wire just before taking readings, after a preliminary run had shown the probable values. Half a minute or more

* It was satisfactory to find the balance independent of whether the wire was moving or not.

was required in taking a final reading, largely because the steady swirl of liquid could not be attained in less time. At times conditions were obtained which were practically steady (water for instance), but at others there was sufficient suspended material to give a slight drift. For instance, on one occasion with No. 1 oil and the 6 mil wire there was a drift equivalent to about a $\frac{1}{2}$ per cent. change in heat loss in one minute, and no further change in the next two minutes. Repeated experiments on this occasion indicated that results for all temperatures of the wire were certainly reproducible to within ± 2 per cent., and usually variations were of the order of ± 1 per cent. or less. If the difficulties increased appreciably, refiltering was resorted to and an improvement thus effected.

(vii.) *Measurement of Velocity.*

No difficulty was experienced in determining the velocity of the rotating arm. The periods for one revolution varied from 3 seconds to 25 seconds. An electrical buzzer recorded each revolution, and it was usually sufficient to time with a stop-watch a convenient number of revolutions. A chronograph arrangement connected to a standard clock was available, but was not necessary for general use. A stroboscopic arrangement could have been used for observing the constancy of the actual speed of the motor during a complete revolution of the arm, but was not required, the galvanometer spot being sufficient indication of the state of affairs.

To obtain the velocity of the wire relative to the liquid, it was necessary not only to measure the velocity of the rotating arm, but also to evaluate the correction for the general circulation of the fluid through which the wire moved.

To measure the swirl velocity the test wire was removed from the rotating arm and mounted vertically in a fixed position in the centre of the trough, so that a dummy wire attached to the rotating arm passed within a millimetre or so from it. Heat losses from the wire in this position were measured with the arm rotating at various speeds, and compared with the losses experienced when it was mounted upon the rotating arm itself. The method of deducing the swirl velocity from the results is as given below.

For a given temperature excess, whether the wire is mounted on the rotating arm or is fixed in the centre of the trough, the heat emitted from it depends only upon the motion of the wire relative to the stream of liquid.

the swirl values obtained refer to the general circulation in the centre of the trough.

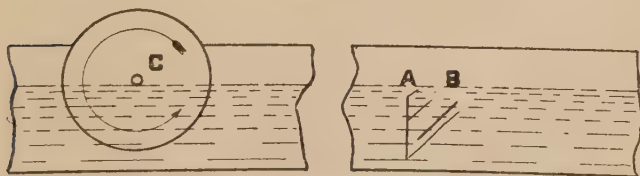
The method adopted as a check on the wire method was not so convenient, but it revealed visually the nature of the flow. The arm was set rotating in the usual way, and when the circulation of liquid in the trough was steady observation was made of the motion of a speck of floating or suspended dust. For some cases tobacco ash was found to be very convenient. The swirl velocity measured was the general drift. The timing of the speck was commenced sufficiently long after the passage of the test wire for the flow to be normal, and was completed just before the returning wire brought the local disturbances round again. Simultaneous observation of floating specks and suspended particles confirmed that there was a central core of liquid $\frac{1}{2}$ in. to $\frac{3}{4}$ in. wide, through which the test wire was moving, where the velocity was practically uniform and independent of depth.

The agreement between these check values and those obtained by the wire method was quite good, as may be seen from Table I., though generally the speck method gives higher results. It will be seen that generally the mean value taken is that given by the wire, although some attention has been paid to the other results. Divergences from the mean are not very systematic, and appear to have an effect on the true velocity of the order of ± 2 per cent.

(viii.) *Velocity of the Natural Convection Current from the Wire.*

Experiments were carried out to ascertain to what extent the natural convection current from the heated wire was comparable with the velocity of the wire relative to the liquid.

Fig. 4.



Arrangements for determining the Velocity of the Natural Convection streams from the hot wire A.

The principle of the method of measuring the velocity of the natural convection current is simple, and is illustrated in fig. 4. The heated wire was mounted vertically at a position A in the trough. A steady circulation of liquid

past the wire was maintained by means of the paddle-wheel C, placed as far away as possible. Convection currents from the wire could then be observed as apparently straight filaments adherent to the wire and trailing behind it at an angle to the vertical depending upon the velocity of the liquid in the trough. Observation was made of the stream velocity at which the filaments were inclined to the vertical at an angle of 45° . In this case, heated particles of liquid were moving so that the vertical component of velocity, due to natural convection, was equal to the horizontal component, which was known from the observed liquid speed.

Errors of refraction in the estimation of the angle of 45° were avoided as follows. A separate piece of wire B was suitably attached to the apparatus, so that it was immersed in the liquid and extended upwards behind the heated wire at this required angle. Speed adjustments were made until the fluid filaments were parallel to it.

These apparent filaments were of course due to the altered refractive index of the liquid which had been heated by the wire. No difficulty was experienced in seeing them with some of the coils. However, with water practically nothing could be detected; so in such cases a speck of soluble dye was attached to the wire at one point. It was quite easy to attach such a speck when desired, introducing it on the end of another piece of wire.

Only the order of the effect was desired, so elaborate arrangements were not adopted. However, by using a transparent trough with plane sides, photographs could be taken and greater precision attained. The results might be of some interest apart from their present application.

The velocities of the natural convection current were measured for water, No. 2 oil, and paraffin. They were more or less proportional to the temperature excess of the wire, and were respectively 0.01, 0.006, and 0.02 cm. per sec. per degree temperature excess of the wire.

The maximum temperature excess used in the main experiments was about 70°C ., whence natural convection streams of the order of $\frac{1}{2}$ to $1\frac{1}{2}$ cm. per sec. are to be expected. These should cause but little error in estimating the velocity of the fluid relative to the wire, for being vector quantities directed at right angles to the motion of the wire, they would only increase the slowest speed (10 cm. per sec.) by about 1 per cent. At higher speeds the effect should be quite negligible.

4. THE LIQUIDS USED.

(i.) *Their nature and condition.*

The cooling power of five liquids was studied. These were distilled water, paraffin, and three transformer oils, of which the most viscous resembled olive oil at 15°C . and became practically solid at 8° to 10°C . Thus a wide range of viscosity (and of k/ν) was covered, and the fluids had the advantage of being of kinds frequently used for cooling purposes. Transformer oils are very carefully refined products. Special care had been taken to ensure that they were dry, clean, and unmixed with other oils or samples, and in a condition satisfying the stringent requirements of their normal use.

Reference has already been made to the care taken to remove fibres of fluff and suspended particles. It was also observed that innumerable bubbles filled the oils when they were freshly poured into a vessel. These would have caused errors in some of the tests, so the liquids were poured carefully and time was allowed for the bubbles to disappear. This delay also allowed the temperature to settle down.

(ii.) *Physical Properties of Liquids used.*

The following outlines the various methods used in measuring for these liquids the physical properties concerned in convective cooling. The results obtained are given in Table II.

(a) * *Density (ρ).*

Determinations of density were made at temperatures of about 15°C ., 50°C ., and 80°C ., an ordinary specific gravity bottle being used. Values for water were taken from tables.

The coefficients of expansion, of importance in natural convection, may be deduced from these results.

(b) * *Viscosity (η) and Kinematical Viscosity ($\nu = \eta/\rho$).*

Determinations of viscosity were made at various temperatures. Values for water were taken from published tables. For the more viscous oils a Redwood (No. 1 type) viscometer was used in accordance with customary practice at the National Physical Laboratory †. Essentially the method consists in determining the time taken for 50 c.c.s. of the oil at the test temperature to flow out of an oil cup through a

* Mr. A. R. Challoner assisted with these observations.

† Higgins, 'N. P. L. Collected Researches,' xi. pp. 1-16.

suitable capillary jet, the initial level of the oil having been accurately adjusted to a certain standard position. In this way "kinematical viscosity" is determined, from which the other coefficient may be calculated.

TABLE II.
Physical constants of Liquids used.

Liquid.	Temp. ° C.	ρ .	$10^4 k$.	η .	ν .	$s\eta/k$ $=c\nu/k$.	s .	c .
Water	12	1.00	14.0	0.0124	0.0124	8.86	1	1.00
	15	1.00	14.0	0.0114	0.0114	8.15		1.00
	20	0.998	14.0	0.0101	0.0101	7.2		1.00
	40	0.992	14.7	0.0066	0.0066	4.48		1.00
	50	0.988	15.0	0.0055	0.0056	3.67		0.99
	75	0.975	...	0.0038	0.0039	...		0.98
	80	0.972	...	0.0036	0.0037	...		0.97
Paraffin.....	12	0.814	3.0	0.0220	0.0270	38.1	0.52	0.42
	15	0.812	3.0	0.0208	0.0256	36.0		0.42
	20	0.809	3.0	0.0189	0.0234	32.6		0.42
	50	0.787	2.8	0.0110	0.0140	20.4		0.41
	60	0.780	2.7	0.0096	0.0121	18.5		0.41
	75	0.769	2.6	0.0078	0.0101	15.6		0.40
	80	0.765	...	0.0075	0.0096	...		0.40
Transformer Oil No. 1	15	0.844	3.2	0.140	0.166	205	0.47	0.40
	20	0.841	3.2	0.116	0.138	170		0.40
	50	0.821	3.1	0.046	0.056	70		0.39
	60	0.814	3.1	0.036	0.044	54		0.38
	75	0.804	3.1	0.027	0.033	40		0.38
	80	0.801	...	0.026	0.032	39		0.38
Transformer Oil No. 2	12	0.876	2.9	0.442	0.505	732	0.48	0.42
	15	0.875	2.9	0.384	0.438	633		0.42
	20	0.872	2.9	0.303	0.347	500		0.42
	50	0.852	2.9	0.090	0.106	149		0.41
	60	0.846	2.9	0.063	0.074	104		0.41
	75	0.836	2.85	0.041	0.049	69		0.40
	80	0.833	...	0.035	0.042	...		0.40
Transformer Oil No. 3	12	0.918	3.0	0.981	1.07	1500	0.46	0.42
	15	0.915	3.0	0.745	0.816	1142		0.42
	20	0.912	3.0	0.506	0.555	776		0.42
	50	0.894	2.9	0.121	0.135	191		0.41
	60	0.888	2.9	0.084	0.095	134		0.41
	75	0.879	2.9	0.059	0.067	93		0.40
	80	0.876	...	0.052	0.060	83		0.40

For paraffin, the Redwood instrument is not suitable, as the flow would be turbulent. In this case the viscosity was measured by comparison with water in an Ostwald viscometer*. This consists of a marked U-tube, the connecting

* I am indebted to Dr. G. Barr, of the Dept. of Metallurgy and Metallurgical Chemistry, for this determination.

piece being of capillary dimensions. The liquid under test stands initially at different levels in the two limbs, and the time is noted which elapses while the liquid meniscus passes certain marks on the tube.

It is desirable to notice (see Table II.) that very considerable viscosity changes arise when the temperature of a liquid is altered.

(c) *Specific Heat (s) and (c).*

The specific heat (s) determined for each oil was the mean value over the range 20°C. to 100°C. It was obtained by the method of mixtures, a measured quantity of the oil under test at a temperature of 100°C. being poured into a calorimeter containing a standard oil at 20°C. The specific heat of the oil chosen as standard had been determined at air temperatures by introducing a copper block at 360°C. into a measured quantity contained in a calorimeter. The standard oil was found to have a specific heat at air temperatures indistinguishable from its mean value over the range 20° to 100°C. Values obtained were reproducible to an accuracy of about 2 per cent.

The capacity for heat (c) per unit *volume* of a liquid is given by the equation $c = sp$.

(d) *Thermal Conductivity (k).*

Thermal conductivities of the fluids were determined by means of an apparatus similar to that described by Goldschmidt*. The method consists of stretching a fine platinum wire along the axis of a small horizontal silver tube full of the test liquid, and measuring the electric current necessary to maintain a measured temperature difference between the wire and the walls of the tube. This gives a relative value for the conductivity of the oil, for it appears that with fine wires, small tubes, and but slight heating the disturbing effects of convection currents are negligible. This is not surprising since the cooling of a fine wire immersed horizontally in a bath of fluid tends to depend only upon the conductivity of the fluid in these conditions.

In the actual apparatus the silver tube, 2 mm. diameter and 12.7 cm. long, was fitted tightly into a massive copper cylinder of 5 cm. diameter, so that the temperature was uniform throughout its length. A 4 mil platinum wire, with current and potential leads attached to its ends, was adjusted

* Goldschmidt, *Phys. Zeits.* xii. p. 417 (1911).

to pass axially through the tube. Since the current leads were helical copper springs they prevented sagging of the test wire when heated. The apparatus* was completely immersed in a quantity of the liquid to be examined, which thus filled the capillary tube.

Measurements were made of the heat dissipation associated with various small temperature excesses of the wire, ranging from 1° or 2° C. to 20° C. For this the Kelvin bridge was used in the manner already described. However, since only slight heating was used, the initial current was reduced to 0.01 ampere so that its heating effect should be negligible in these new conditions. The currents used for heating ranged up to about 1 ampere.

Thermal conductivities were calculated on the assumption of radial flow of heat between the wire and the inner surface of the silver tube. For moderate heating results so calculated increased as the temperature excess was raised, showing that convection currents had become appreciable. For slight heating, however, results were practically constant. The conductivity given in the tables is the value to which results tended as the temperature excess was reduced to zero.

The experiments were carried out with the liquids at two temperatures, about 20° C. and 80° C. respectively. No allowance was made for end effects, but probably the length of wire used (12.7 cm.) was sufficient to render these of little importance.

(e) *The ratio (k/cv).*

On the kinetic theory of gases the thermal diffusivity (k/c) bears a ratio to the kinematical viscosity \dagger (ν), which is constant for a given kind of gas, and depends upon the number of atoms in a molecule. For liquids, owing to complexity of structure, such constancy does not hold, and a wide range of value is covered. It is to be expected on general grounds that this fundamental ratio between the factors determining the fluid flow and the heat transfer will be important in determining the cooling power of fluid streams. Consequently, for each of the liquids, k/cv has been evaluated for a series of temperatures. It is the factor determining the complexity of a fluid when thermal and viscous actions are involved together, *i. e.* in convection of heat.

* This apparatus was available, having been constructed previously by Dr. Ezer Griffiths.

\dagger Diffusivity of momentum.

5. THE WIRES USED.

The wires used were of specially pure platinum, and were of three sizes, being about 4, 6, and 8 mils* in diameter. Mean details are given in Table III.

TABLE III.
Details of the Wire used.

Total length between current terminals = 6.08 cm.
Mean length between potential leads = l cm.
Resistance per unit length at 0° C. = R_0/l ohms per cm.

Wire.	Diam.	l .	R_0/l .	α .
4 mil	0.0102 cm.	4.8	0.118	0.00386
6 mil	0.0152	*4.75	0.0564	0.00380
8 mil	0.0204	4.98	0.0294	0.00387

* Lengths 3.15 and 6.08 cm. were used in the end correction test.

Potential leads of 4 mil platinum wire were welded to the test wire at a distance from its ends. The length of wire between these leads was easily measured. Diameters were measured by means of a micrometer screw gauge reading to one ten thousandth part of an inch.

Before the wires were cut up into lengths for mounting in the holder of the rotating arm, temperature coefficients of resistance were determined for annealed samples. Lengths of about 30 cm. were used, and resistances were determined at the temperatures of melting ice, and of the vapour of water boiling under normal atmospheric pressure. Sulphur points had previously been determined. For these measurements use was made of a Callendar and Griffiths resistance bridge reading to 0.0001 ohm. However, in one case, using the Kelvin bridge arrangement, a check measurement was obtained for a portion of wire mounted in the normal manner in the holder of the whirling arm. The wire was immersed in a stirred oil bath at a few fairly steady temperatures, and the temperature coefficient of resistance thus obtained agreed to within 2 per cent. with the more accurate determination previously made.

During cooling experiments with the arm rotating, the finer wires tended to stretch slightly, presumably owing to

* 1 mil=0.001 inch.

the strains set up. This was most readily detected by the change in resistance of the wire at 0°C. , a quantity which was regularly calculated from the measured resistance at the temperature of the oil. Necessary allowances for such changes were made in the heat-loss calculations. The effect, however, was slight, and in the whole life of a wire did not amount to more than 1 per cent. increase in length.

Throughout the experiments care was taken to keep the wire bright and free from adherent fluff, etc. If by accident it was over-heated, it became blackened with carbonized oil. On these occasions the wire was removed from the liquid, and the deposit was burnt off by heating the wire electrically to a dull redness. Naturally, after this procedure, particular attention was paid to alterations in resistance.

(i.) *Calculation of the heat-loss from the wire.*

The heat-loss from the wire during an experiment was calculated from the current " i " flowing through it, and from its hot resistance R , as measured on the Kelvin bridge. Results were converted from watts to calories per second, using 4.184 as the conversion factor. They were expressed in the form of heat-loss per cm. length per degree temperature elevation.

(ii.) *Calculation of the temperature excess of the wire.*

The temperature excess of the wire was calculated from the increment of resistance dR : on the platinum scale it is given by $\theta = dR/R_0\alpha$, where R_0 is the resistance of the wire at 0°C. , and α is the temperature coefficient of resistance. Corrections to the centigrade scale involve increasing the above value by about 1 per cent. for values of θ up to 20°C. and by $1/5$ per cent. for $\theta = 60^{\circ}\text{C.}$ Such corrections are almost negligible, being greatest for the low values of θ , where the experimental errors were greatest. They may be obtained conveniently from tables showing the relation between the two scales*.

(iii.) *Correction due to radial temperature gradient in the wire.*

There is also a small correction due to the fact that, through its immediate proximity to the liquid, the surface of the wire becomes slightly cooler than the core. Calculation on certain assumptions shows the temperature difference

* Harker, Phil. Trans. A. cciii. p. 343 (1904).

between the core of the wire and its surface to be proportional to the rate of surface cooling*. In the notation of this paper it is given by $H\theta/4\pi K$, where K ($=0.166$ cal. per cm. per degree) is the thermal conductivity of the platinum of which the wire is composed. The mean temperature of the wire as obtained by resistance measurements is thus in excess of the surface temperature by an amount $d\theta$ given by $d\theta/\theta = H/8\pi K = 0.24 H$. From the tables of H , given later, we find that for water the relative error $d\theta/\theta$ is appreciable only for the highest speeds, and that for all other liquids it is invariably less than $1/3$ per cent. No correction for this effect has therefore been made.

(iv.) *Correction for conduction by the potential leads and terminals.*

The wire was clamped to the holder by two terminals about 6.1 cm. apart, and the potential leads (4 mil) were attached at about $\frac{1}{2}$ cm. from the clamped ends. It was considered that at this distance the terminals would have no cooling effect on the test portion of the wire, and that the potential leads were sufficiently thin to be negligible in this respect.

If the terminals exert any appreciable cooling, measured heat-losses should depend upon the distance of the potential leads from the ends. With the liquids no such dependence was found. Direct test with oil No. 1 gave practically identical results, whether the potential leads were attached to the wire at the clamping terminals, at $\frac{1}{2}$ cm. from them, or at double this distance. Even with air as the fluid (where end cooling would be of much greater relative importance owing to the much smaller rates of surface cooling) differences were not very systematic and were limited to ± 5 per cent.†

It was inferred from these experiments that the heat lost by conduction to the terminals and potential leads was negligible in comparison with that carried off by the liquid streams.

6. RESULTS.

(i.) *Tables and Curves showing heat-losses obtained with each of the liquids.*

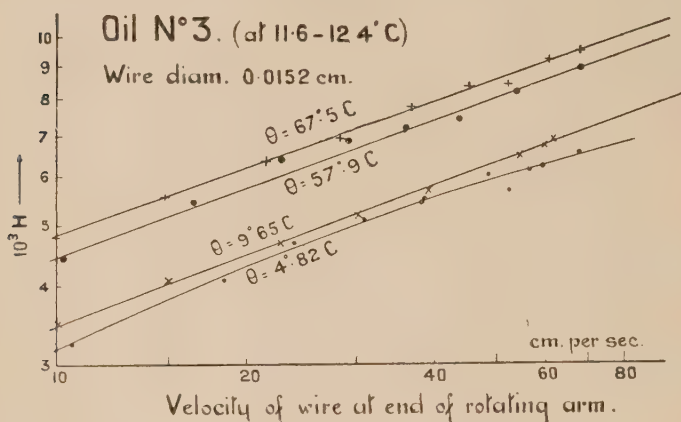
The results obtained in the experimental work were expressed in terms of the heat-loss H per centimetre length of wire per degree centigrade temperature excess. Observations were taken for various velocities of rotation, and for a series of values of temperature excess.

* King, *loc. cit.*

† Greatest values were indicated when the potential leads were attached at the extreme ends of the wires.

To present results in convenient comparative tables, it was necessary to obtain by interpolation values appropriate to certain selected velocities and temperatures. In this connexion a graph is given, fig. 5, which shows experimental points obtained for one of the oils, and the smooth curves from which representative results were read off. As regards

Fig. 5.



Specimen Experimental Results.

temperature excess, it will be seen that the values deviated but slightly from a convenient series, viz. 5° , 10° , 60° , and 70°C . The small adjustments necessary to give heat-losses corresponding to these exact values were obtained by further curves. The procedure with this oil is typical.

Tables IV. (i.–v.) give results compiled in the above

TABLE IV.—Heat losses ($10^5 H$) from Wires in Liquid Streams.
(H =calories per cm. per sec. per $^\circ\text{C}$. temperature excess of wire.)

(i.) Distilled Water *.							
Diameter of Wire.	Mean temperature of water.	Temperature excess of Wire.	$10^5 H$. For values of relative velocity (cm. per sec.).				
			10.	20.	40.	60.	70.
cm.	$^\circ\text{C}$.	$^\circ\text{C}$.					
0.0204	11	8	2070	2900	4060	4950	5340
		20	2200	3090	4300	5250	5650
		35	2560	3460	4800	5830	6300

* Worthington and Malon give a value $10^5 H = 2960$, when $d = 0.0256\text{ cm.}$, $\theta = 30^\circ\text{C.}$, and $v = 14.85\text{ cm. per sec.}$ Frankl. Inst. J. 184. p. 115, 1917).

Table IV. (*continued*).

(ii.) Paraffin Oil.

Diameter of Wire.	Mean temperature of oil.	Temperature excess of Wire.	10 ⁵ H. For values of relative velocity (cm. per sec.).				
			10.	20.	40.	60.	70.
cm.	° C.	° C.					
0.0152	17.6	5	460	622	840	993	1055
		50	487	677	940	1140	1230
0.0204	11	5	500	697	978	1190	1280
		50	572	798	1110	1350	1455

(iii.) Transformer Oil No. 1.

0.0152	16	5	384	515	693	825	875
		10	400	532	705	827	880
		20	428	570	757	890	950
		40	450	603	795	944	1010
		60	467	625	835	1000	1068
0.0204	15	5	480	644	855	1014	1076
		10 †	483	646	863	1024	1095
		20	505	684	910	1070	1138
		40	550	745	990	1160	1237
		60 ‡	588	786	1040	1225	1305

(iv.) Transformer Oil No. 2.

0.0102	11	5	315	425	553	630	660
		50	383	512	688	814	870
0.0152	11	5	358	468	610	713	758
		10	382	492	650	765	812
		20	395	520	687	808	860
		40	430	565	740	870	923
		60	458	598	778	906	960
0.0204	14	5	440	578	760	890	944
		50	542	727	972	1155	1230

(v.) Transformer Oil No. 3.

0.0152	12	5	335	450	568	640	660
		10	363	468	600	695	737
		60	474	600	765	880	930
0.0204	16	5	406	535	702	825	877
		10	417	548	720	847	900
		20	468	612	800	940	1000
		40	512	672	890	1050	1118
		60	560	738	970	1140	1210

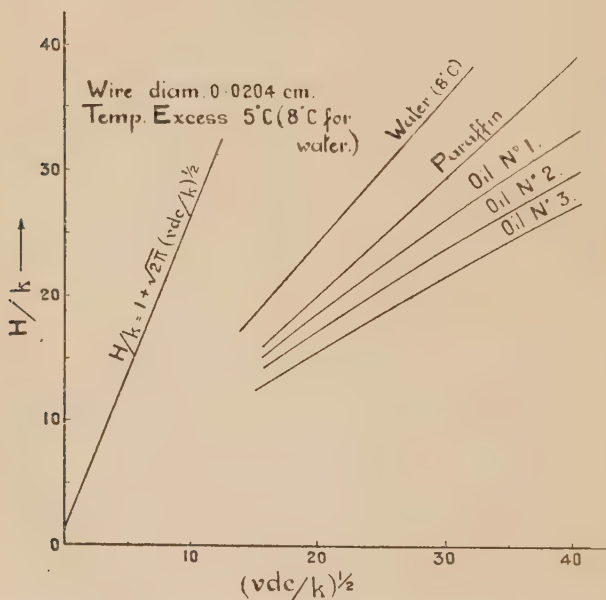
† A series at 10° C. excess taken much earlier and with a different wire gave smoothed results indistinguishable from these at the lower velocities, and falling to about 3 per cent. lower at 70 cm. per sec.

‡ An earlier series at 60° C. with a different wire gave smoothed values indistinguishable from these.

manner, and the values given bear a very close relation to actual points obtained in the experimental work. They present in some 200 orderly values the results of four or five times this number of observations.

A trial was made of representing these results on graphs in which heat-loss was plotted as ordinate against the square root of the velocity as abscissa. In general, the lines obtained were slightly curved, so that the results could not be conveniently expressed by a formula similar to that found

Fig. 6.



Results for Viscous Liquids compared with formula for Inviscid Fluids.

for gases (equation 2). Fig. 6 shows a series of results plotted on a graph of this form.

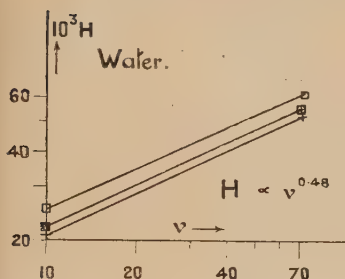
On plotting logarithmically, however, excellent straight lines were usually obtained, and results were studied from this point of view. Fig. 7 shows families of curves obtained for each of the liquids used. It is immediately evident that for a given liquid the lines are practically parallel, the slope having no marked dependence upon the diameter of the wire or upon its temperature excess. There is possibly a tendency

for the slope to be greater for the larger wires, but otherwise occasional irregularity of slope or of curvature appears to be

Fig. 7.

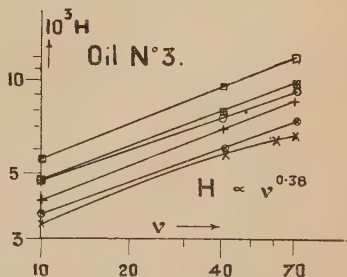
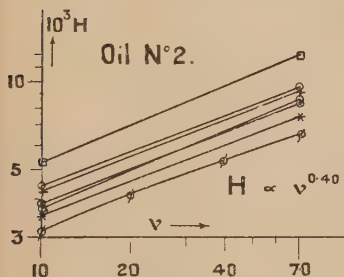
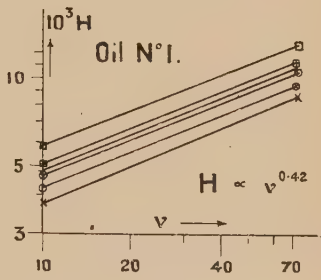
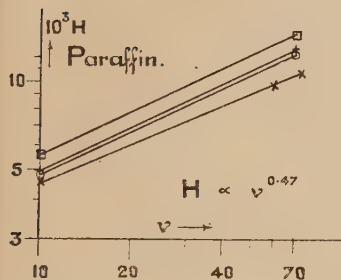
HEAT LOSS - VELOCITY CURVES.

H = Heat loss per cm per degree Temperature excess. (calories)
 v = Velocity of wire relative to fluid. (cm. per sec.)



Key

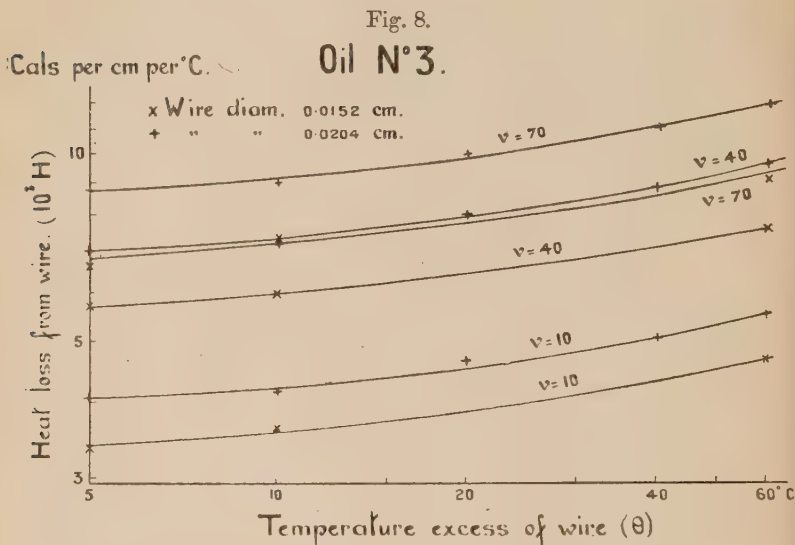
Wire diam cm.	Temp. excess θ		
	Low	Med.	High
0 0102	\emptyset		ϕ
0 0152	x	\otimes	o
0 0204	+	\boxplus	\square



due to experimental error, for with different liquids it is not associated consistently with any particular set of conditions.

Parallel lines are also obtained on plotting results with temperature excess (θ) as abscissa. This may be seen from

the curves shown in fig. 8. Thus the effect of temperature is to introduce a factor dependent only upon the nature of the liquid used.



Specimen Curves. Heat Loss—Temperature Excess.

(ii.) *Empirical relations.*

If we accept the families of curves of figs. 7 and 8 to be parallel lines, rectilinear in the former case, we find the following general formula covers the results:—

$$H = Mv^n(1 + a\theta + b\theta^2), \quad (9)$$

where n , a , and b depend only upon the liquid used, while M depends also upon the diameter of the wire.

The values of " a " and " b " are given later in Table V. Undoubtedly they are intimately connected with the rate at which the physical properties of the fluids vary with temperature. Since our theoretical formulæ are restricted to cases where the physical properties remain constant, discussion will be devoted mainly to cases where the temperature excess " θ " is so small that changes in the properties of the fluids are inappreciable. In these circumstances equation (9) becomes

$$H = Mv^n. \quad (10)$$

¶ Trial of a more general relation of the form *

$$H/k = N(vdc/250k)^n$$

showed N to depend upon both vdc/k and k/cv , so that the effects of these two terms are inter-related. While in the present experiments " n " exhibits no appreciable variation with vdc/k , being given by

$$n \text{ varies as } (k/cv)^{0.03-0.04}, \quad . \quad . \quad . \quad (11)$$

published values for gases cover a much wider range and indicate that

$$n \text{ varies as } (vdc/k)^{0.066}. \quad . \quad . \quad . \quad (12)$$

Now to a certain degree of approximation equations (11) and (12) can be combined in the form

$$n \text{ varies as } \{(vdc/k) \times (k/cv)^{\frac{1}{2}}\}^{0.066}, \quad . \quad . \quad (13)$$

and it was decided to test whether the new variable thus revealed would express completely the inter-relation of (vdc/k) and k/cv .

(iii.) *Formula covering all results.*

It was then found possible to express our experimental results in a formula of the type

$$H/k = A \cdot B \cdot C \cdot F \{(vdc/k) \times (k/cv)^{\frac{1}{2}}\}^{\dagger}, \quad . \quad (14)$$

where $A (= 1 + a\theta + b\theta^2)$ depends upon the temperature excess of the wire and the nature of the fluid, and is undoubtedly connected with the rate at which the physical properties of the fluid vary with temperature.

B represents a small residual effect of wire diameter, additional to that covered by the " vd " term. It depends somewhat upon the nature of the liquid.

$C [= f(k/cv)]$ is dependent only upon the nature of the fluid. It varies as $(k/cv)^{-m}$, where $m=0$ for gases and increases to about 0.19 in the region of the viscous oils.

¶ * In the experiments the minimum value of vdc/k for each liquid was about 250. By writing $vdc/250k$ instead of vdc/k the advantage is gained that N has a definite physical meaning. It is the lowest value of H/k actually measured, instead of being the extrapolated value associated with $vdc/k=1$.

† $\{(vdc/k) \times (k/cv)^{\frac{1}{2}}\} = \{(vdc/k) \times (vd/v)\}^{\frac{1}{2}} = \{(vd/v) \div (k/cv)^{\frac{1}{2}}\}$.

The formula is of the general form expected (3), except for the small corrections represented by the terms A and B. The particular grouping of variables within the brackets correlates the various slopes of the lines in fig. 7 with the curvature observed when study is made of an extended $vd\epsilon/k$ curve for gases.

TABLE V.
Values of A and B.

$A=1+a\theta+b\theta^2$, where θ =temperature excess of wire.

Fluid.	10^3a .	10^5b .	Values of B		
			For wire diam. cm.		
			0.0102	0.0152	0.0204
(Air)*	(0.91-0.94)	(0.95-0.97)	1
Water	2.64	12	1
Paraffin	2.94†	0.98	1
Oil No. 1	7.25	-5.08	...	0.92	1
Oil No. 2	10.7	-8.42	0.93	0.90	1
Oil No. 3	13.5	-9.52	..	0.90	1

* Calculated from King's results.

† Mean value obtained from results at temperature excesses 5° C. and 50° C.

TABLE VI.
Relation between C and k/cv .

k/cv .	cv/k .	C.
1	1	1.02
0.1	10	1.17
0.01	100	1.46
0.001	1000	1.98

(a) *The dependence of A upon temperature variations in the physical properties of the liquid.*

The physical property of the liquid which depended most upon temperature was the viscosity. In some cases the cold oil was as much as ten times as viscous as hot oil at the

temperature of the wire. Such variations may be too great to be covered by a simple correction applied to a theoretical formula based upon assumptions of constancy. The temperature effect has therefore been left in the empirical form given above, k , c , and ν having the values appropriate to the cold fluid.

It is worth notice, however, that the effect is generally rather greater than can be allowed for in the final formula by supposing the physical properties of the fluid to be those appropriate even to the temperature of the hot wire. Generally, for 60° C. temperature excess of the wire, about 10 per cent. increment in heat-loss has still to be accounted for.

(b) *Variation of B with wire diameter.*

By plotting curves showing the relation between H and νd , and noticing the factors B necessary to bring the curves for the finer wires into coincidence with the curves for the stouter, the values given in Table V. were obtained. The total range in the value of B , about 10 per cent., is too small for deducing the laws governing its variations, but a definite diameter effect is indicated. The order of the effect found for liquids is seen to be the same as that for air as calculated from King's formula. Special experiments with the present apparatus, on the air-cooling of the 6 mil and 8 mil wires, gave ratios of heat-loss in good agreement with King's values*. This eliminates experimental error as an explanation of the differences found for these wires when liquids were used.

Considerations of the theory of dimensions show that, in the theoretical formula, wire diameter and velocity must always occur together, unless variables are involved additional to those included above. If free convection effects are included the variable ag may be introduced, where a =volume coefficient of expansion of the fluid and g =acceleration due to gravity. A term $cgd^3a\theta/k\nu$ † (or its equivalent), found by the author to cover free convective cooling

* Results were not corrected for swirl, but comparisons were made at definite arm velocities. The actual magnitudes of heat-loss were almost identical with those calculated from King's formula, provided the swirl velocity in the trough was about 12 per cent. of the arm velocity. Such a figure is reasonable since values of 10 per cent. to 20 per cent. were obtained when liquids were used. King found a 5 per cent. swirl in a large room.

† If, in addition to the free convection variables, the density of the fluid is also separately introduced [it is already included to some extent in $c(=s\rho)$ and $\nu(=\eta/\rho)$] a term results which contains d without either a or ν . Richardson has discussed the relation borne by such a term to King's results (Phys. Soc. Proc. xxxii. p. 405, 1920).

in viscous liquids, may then be expected in the complete equation. If B owes its origin to this new correction term, its dependence (Table V.) upon the nature of the liquid (ca/kv) is thus explained. Some associated dependence upon the temperature excess θ is also to be expected, and this may explain the residual relation between A and θ not previously accounted for. It may also explain why, in King's results, his diameter term is associated with a residual θ term much larger than that which operates upon the velocity group.

While the term B may have its origin in natural convection effects, other possibilities must not be overlooked. As previously stated, the diameter effect indicated by Hughes's results for large cylinders is in the opposite sense to that found in the present experiments with fine wires. The distribution of heat flux across the boundary may have been different in this case.

(c) *Relation between C and the nature of the liquid.*

$$C = f(k/cv) \quad \dots \dots \dots (15)$$

Fig. 9 shows the manner in which C depends upon the value of k/cv for the liquid used. It was obtained for the 8 mil wire and for small values of temperature excess by plotting H/k as ordinate against $\{vdc/k \times (k/cv)^{\frac{1}{2}}\}$ as abscissa. A series of curves was thus obtained parallel to neighbouring portions of the extended curve for air, but displaced by amounts depending upon the nature of the liquid used. Measurement was made of the amounts of displacement, that is, of the factors C necessary to bring all the curves into coincidence with that for air. It is seen that a smooth relation is indicated between C and k/cv . Values read off from fig. 9 are given in Table VI.

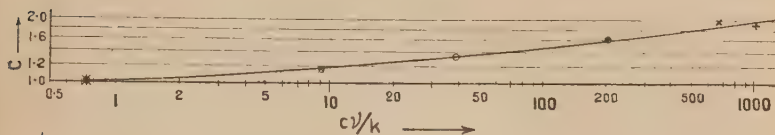
(d) *Curve showing correlation effected.*

Fig. 10 has been drawn to show the extent to which the final formula (14) represents the results of the experimental work. Ordinates give the value of $[H/k] \div [A \cdot B \cdot C]$ and abscissæ represent $[vdc/k \times (k/cv)^{\frac{1}{2}}]$. Values taken for A, B, and C are those given above.

The points shown represent the experimental values for liquids as given in Tables IV. (i.-v.). For convenience only a selected number of points have been plotted, and these refer

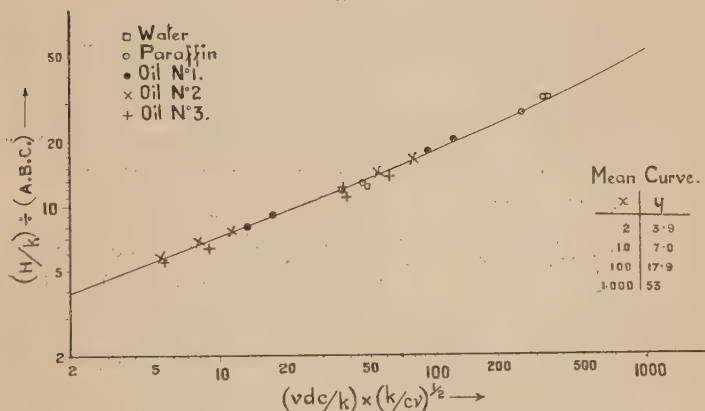
only to small temperature excesses, but each wire is represented by high (70 cm. per sec.) and low (10 cm. per sec.) velocities for each of the liquids used.

Fig. 9.



Relation between C and cv/k .

Fig. 10.



Representative Results for all Fluids plotted on one curve.

The line drawn is part of the mean curve* for gases referred to earlier, re-drawn on the present basis, taking for air $k/cv=1.35$ †. In the region shown, $H/k.A.B.C.$ varies as $[vdc/k \times (k/cv)^{\frac{1}{2}}]^n$, where n varies from 0.38 to 0.48.

* The curve is the mean for gases obtained by the method described elsewhere. No correction terms in B or in $(1+a\theta+b\theta^2)$ have been used, but temperature changes in the properties of air have been approximately allowed for by taking values of k , c , and v for temperatures intermediate between that of the cold air and that of the hot wire. Little difference in the mean curve is obtained if the constants are evaluated for a temperature equal to that of the hot wire.

† c refers to unit volume and constant pressure. If the constant volume value is taken, a slight alteration in the form of the function f is all that is necessary to give a curve practically as good as the present one. All values of $f(x)$ for x greater than 9 have to be reduced by 30 per cent. and the alteration in $f(x)$ for lower values of x falls to zero as x approaches 1.90, the new value of k/cv for air.

The figure shows that equation (14) satisfactorily represents our results. It is as far as possible of the type suggested by theoretical considerations, but is empirical as regards the actual form of the functions given.

I desire to express my thanks to Dr. G. W. C. Kaye, Superintendent of the Physics Department of the National Physical Laboratory, for the helpful manner in which all the facilities for the work have been provided. Mr. R. R. Strand, Senior Observer, helped with the constructional work and rendered very valuable assistance with the experimental observations.

January 1924.

CV. *Note on the Auto-Transformer and on the Impedance as a Tensor.* By W. H. INGRAM, M.Sc.*

SYNOPSIS.—A formula is given for the voltage at a tap in the primary of a transformer, and it is shown that under given conditions this voltage will be independent of the load on the transformer.

Impedance is defined as a tensor.

IN a recently published note † I gave a voltage formula for the step-down auto-transformer which showed that the secondary terminal voltage will be independent of the exciting current when a certain ratio is obtained between the self-impedances of the two windings into which the auto-transformer is divided by the tap. A similar phenomenon may be exhibited in the case of a transformer having a tapped primary or, which is the same thing, a step-down auto-transformer having a (metallically independent) tertiary.

Let the auto-transformer be the one specified in the above-mentioned Note, and the tertiary have n_3 effective turns. The working flux, or the flux that links the windings n_1 , n_2 , and n_3 perfectly, is to be imagined as also linking the winding n_0 of the imaginary core-impedance circuit. The system n_1 , n_2 , n_3 , and n_0 represents an ideal voltage transformer, giving the first of the following relations, the next two being given by the usual current and m.m.f. conditions:

$$\left. \begin{aligned} e_1/n_1 &= e_2/n_2 \text{ etc.}, \\ i_1 &= i_2 + i_a, \\ n_1 i_1 + n_2 i_2 + n_3 i_3 &= n_0 i_0. \end{aligned} \right\} \dots \dots (1)$$

* Communicated by the Author.

† Phil. Mag. xlvii. p. 241 (Jan. 1924).

Applying the usual circuital work theorem :

$$\left. \begin{aligned} E_1 &= e_1 + Z_1 i_1 + Z_{12} i_2 + Z_{13} i_3 + E_a, \\ E_a &= e_2 + Z_2 i_2 + Z_{21} i_1 + Z_{23} i_3, \end{aligned} \right\} \dots (2)$$

where E_1 is the voltage impressed on the transformer primary ; E_a the auto-transformer secondary voltage ; i_3 the current in the n_3 winding ; Z_1 the self-impedance of the winding 1 (the reactance of which is due to the leakage flux exclusively associated with this winding, and the resistance of which is made up of the ohmic resistance of the conductor plus small in-phase effects due to eddys, etc.) ; and Z_{13} , for instance, the mutual impedance between the windings 1 and 3, the reactive part of which is due to flux linkages between the two coils denoted by the subscripts. Solving for E_a in terms of the impressed voltage E_1 , the delivered currents i_a and i_3 , and the exciting current i_0 :

$$\begin{aligned} E_a = \frac{n_2}{n_1 + n_2} E_1 - n_0 & \left(\frac{n_2 Z_1 - n_1 Z_2}{(n_1 + n_2)^2} + \frac{n_2 Z_{12} - n_1 Z_{21}}{(n_1 + n_2)^2} \right) i_0 \\ & - \left[\frac{n_1^2 Z_2 + n_2^2 Z_1}{(n_1 + n_2)^2} - n_1 n_2 \frac{Z_{21} + Z_{12}}{(n_1 + n_2)^2} \right] i_a \\ & - \left[n_3 \frac{n_2 Z_1 - n_1 Z_2}{(n_1 + n_2)^2} + n_3 \frac{n_2 Z_{12} - n_1 Z_{21}}{(n_1 + n_2)^2} \right. \\ & \quad \left. - \frac{n_2 Z_{13} - n_1 Z_{23}}{(n_1 + n_2)} \right] i_3^* \dots (4) \end{aligned}$$

The coefficient of i_3 simplifies somewhat from the fact that $Z_{13} : Z_{23} \sim M_{13} : M_{23} = n_1 : n_2$ (since the paths of all the stray fluxes in any actual iron-cored transformer are mainly iron-free, $Z_{13} i_3$ is practically equal to $M_{13} di_3/dt$, numerically). The impedances Z_{12} , Z_{21} (equal) have no significance, and should be omitted in any case where there is no tertiary winding, as it is here introduced merely to take care of the flux that links the windings 1 and 2 but not 3, which flux is to be considered as part of the working flux when there is no tertiary.

From the fact that Z_1 and Z_2 may represent the self-impedances *plus* any external impedances that may be introduced in series with the windings 1 and 2 respectively,

* The sign of i_3 here is taken to conform with the actual direction of the current. Formula (4) also applies when the winding n_3 is an auto-transformer booster winding ; when the tertiary is connected as a negative booster, the sign of n_3 is to be reversed.

it is clear that by a manipulation of these latter impedances the coefficient of i_3 may be reduced to null. The line and generator impedances may also be included in Z_1 , in which case E_1 becomes the internal e.m.f. of the generator.

The present opportunity seems not inappropriate to point out that impedances are *tensors*, and when defined as such enable us to put the "symbolic" method on a purely vector basis. The imaginary number unit occurring in the usual $Z = R + \sqrt{-1}X$ is entirely devoid of any physical significance, serving merely to establish a certain orthogonality, and has a further disadvantage in that the coordinate system must be adjusted to every i upon which Z operates.

The tensor components of Z may be obtained from the work theorem for the most general simple circuit having lumped constants :

$$E = Ri + L \frac{di}{dt} + \frac{1}{c} \int i dt. \quad . \quad . \quad . \quad (5)$$

If g_1 be a unit vector in the reference-line (see the first paragraph of the above-mentioned Note) we have :

$$\left. \begin{aligned} i \cdot g_1 &= i_{\max} \cos (i, g_1) = i, \\ E \cdot g_1 &= E, \end{aligned} \right\} \quad . \quad . \quad . \quad (6)$$

which may be substituted in (5), giving :

$$E = \left[R + \left(L\omega - \frac{1}{\omega C} \right) \omega_1 \times \right] i = [R + X \omega_1 \times] i, \quad (7)$$

where ω_1 is a unit vector perpendicular to the plane containing i and g_1 , ω the angular velocity of the vector E . In obtaining (7) the relations

$$\frac{di}{dt} = \omega \times i, \quad \int i dt = \frac{-1}{\omega} \omega_1 \times i \quad . \quad . \quad . \quad (8)$$

are made use of, the first of which is obtained directly from elementary vector considerations, and the second by an "educated guess" and verified by differentiation.

An n -fold iteration of Z is written Z^n , and from geometrical considerations is a tensor that rotates n times the angle Z does and stretches to the n th power of the stretch of Z . The reciprocal tensor Z^{-1} is equal to Z_c/Z^2 , where Z_c is the conjugate ($R - X \omega_1 \times$) of Z , and $Z^2 = R^2 + X^2$. The distributive law and the commutative and associative laws of addition and multiplication for operator and operand are the same as for

If we substitute these expressions in (2), we obtain the energy function V as a quadratic function of the coefficients $\alpha_1, \alpha_2, \dots$

By variation of these coefficients we obtain

$$\delta V = \int (\delta X_\nu u + \delta Y_\nu v) ds, \quad . \quad . \quad . \quad (5)$$

where $\delta X_\nu, \delta Y_\nu$ are the corresponding variations of boundary tensions, and u, v are the boundary displacements.

When the right side of (5) is equal to zero, we obtain

$$\delta V = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

To obtain an approximate solution of any problem, we take a few terms in the series (3), and, using the conditions (6), determine the coefficients $\alpha_1, \alpha_2, \dots$ from the linear equations

$$\frac{dV}{d\alpha_1} = 0, \quad \frac{dV}{d\alpha_2} = 0, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Making use of (2) and (4), we may write these equations as follows :

$$\begin{aligned} \iint \{ & (\phi_{0yy} + \Sigma \alpha_m \phi_{myy}) \phi_{nyy} + (\phi_{0xx} + \Sigma \alpha_m \phi_{mxx}) \phi_{nxx} \\ & - \sigma (\phi_{0yy} + \Sigma \alpha_m \phi_{myy}) \phi_{nxx} - \sigma (\phi_{0xx} + \Sigma \alpha_m \phi_{mxx}) \phi_{nyy} \\ & + 2(1 + \sigma) (\phi_{0xy} + \Sigma \alpha_m \phi_{mxy}) \phi_{nxy} \} dx dy = 0. \end{aligned} \quad (8)$$

In many cases the stress-distribution is independent of the elastic constants. We can then put $\sigma = 0$, and replace equation (8) by a simpler equation as follows :

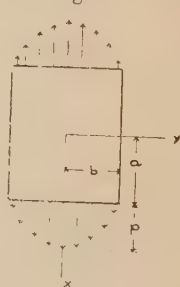
$$\begin{aligned} \iint \{ & (\phi_{0yy} + \Sigma \alpha_m \phi_{myy}) \phi_{nyy} + (\phi_{0xx} + \Sigma \alpha_m \phi_{mxx}) \phi_{nxx} \\ & + 2(\phi_{0xy} + \Sigma \alpha_m \phi_{mxy}) \phi_{nxy} \} dx dy = 0, \end{aligned} \quad (8)'$$

§ 2. As a first example of the method outlined, we proceed to consider the stress-distribution in a rectangular plate (fig. 1), on the edges ($x = \pm a$) of which known normal tractions act.

We choose the function ϕ_0 in such a manner as to satisfy the boundary conditions. The functions ϕ_1, ϕ_2, \dots we take in such a form that the corresponding stresses vanish at the boundary.

For example, in the case of a parabolic distribution of tensile tractions on the edges $x = \pm a$ we may assume a

Fig. 1.



stress-function of the type :

$$\phi = \frac{1}{2}py^2\left(1 - \frac{1}{6}\frac{y^2}{b^2}\right) + (x^2 - a^2)^2(y^2 - b^2)^2(\alpha_1 + \alpha_2x^2 + \alpha_3y^2 + \dots). \quad (9)$$

To the first member of the expression corresponds the parabolic stress-distribution represented in fig. 2 by the curve I. If we take two first members of (9), we have to calculate the coefficient α_1 . Using in this case equation (8), we obtain

$$\alpha_1\left(\frac{64}{7} + \frac{256}{49}\frac{b^2}{a^2} + \frac{64}{7}\frac{b^4}{a^4}\right) = \frac{p}{a^4b^2}.$$

In the case of a square plate ($a=b$) we obtain

$$\alpha_1 = 0.04253 \frac{p}{a^6}.$$

The stress-components will be

$$X_x = p\left(1 - \frac{y^2}{a^2}\right) - 0.1702 p\left(1 - \frac{3y^2}{a^2}\right)\left(1 - \frac{x^2}{a^2}\right)^2,$$

$$Y_y = -0.1702 p\left(1 - \frac{3x^2}{a^2}\right)\left(1 - \frac{y^2}{a^2}\right)^2,$$

$$X = -0.6805 p \cdot \frac{xy}{a^2}\left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{y^2}{a^2}\right).$$

The corresponding distribution of stresses X_x on the cross-section $x=0$ is represented by the curve II. (fig. 2).

To obtain a closer approximation, we take the first four members of (9). Then the equations for the calculation of α_1 , α_2 , and α_3 will be

$$\left. \begin{aligned} \alpha_1\left(\frac{64}{7} + \frac{256}{49}\frac{b^2}{a^2} + \frac{64}{7}\frac{b^4}{a^4}\right) + \alpha_2a^2\left(\frac{64}{77} + \frac{64}{49}\frac{b^4}{a^4}\right) \\ + \alpha_3a^2\left(\frac{64}{49}\frac{b^2}{a^2} + \frac{64}{77}\frac{b^6}{a^6}\right) &= \frac{p}{a^4b^2}, \\ \alpha_1\left(\frac{64}{11} + \frac{64}{7}\frac{b^4}{a^4}\right) + \alpha_2a^2\left(\frac{192}{143} + \frac{256}{77}\frac{b^2}{a^2} + \frac{192}{7}\frac{b^4}{a^4}\right) \\ + \alpha_3a^2\left(\frac{64}{77}\frac{b^2}{a^2} + \frac{64}{77}\frac{b^6}{a^6}\right) &= \frac{p}{a^4b^2}, \\ \alpha_1\left(\frac{64}{7} + \frac{64}{11}\frac{b^4}{a^4}\right) + \alpha_2a^2\left(\frac{64}{77} + \frac{64}{77}\frac{b^4}{a^4}\right) \\ + \alpha_3a^2\left(\frac{192}{7}\frac{b^2}{a^2} + \frac{256}{77}\frac{b^4}{a^4} + \frac{192}{143}\frac{b^6}{a^6}\right) &= \frac{p}{a^4b^2}. \end{aligned} \right\} (10)$$

In the case of a square plate we obtain

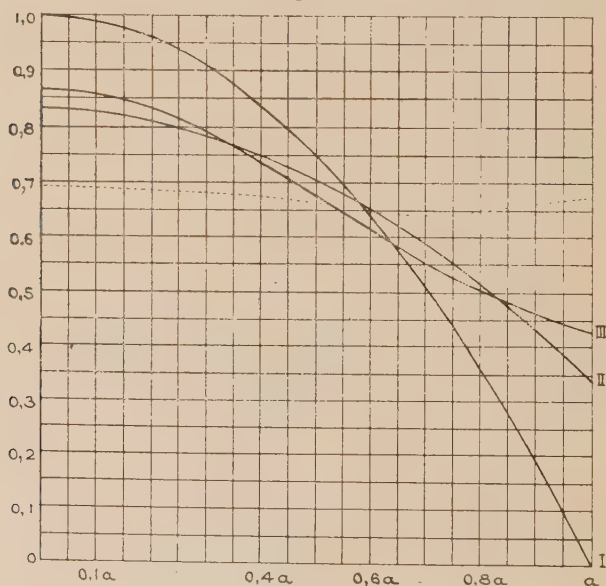
$$\alpha_1 = 0.04040 \frac{p}{a^6}; \quad \alpha_2 = \alpha_3 = 0.01174 \frac{p}{a^8}.$$

The stress-distribution on the cross-section $x=0$ will be given by

$$(X_x)_{x=0} = p \left(1 - \frac{y^2}{a^2} \right) - 0.1616 p \left(1 - \frac{3y^2}{a^2} \right) + 0.0235 \left(1 - 12 \frac{y^2}{a^2} + 15 \frac{y^4}{a^4} \right).$$

On fig. 2 it is represented by the curve III.*

Fig. 2.



With an increase in the length of the plate, the stress-distribution over the cross-section $x=0$ becomes more and more uniform. If we take for instance $a=2b$, we have from the equations (10)

$$\alpha_1 = 0.07983 \frac{p}{a^4 b^2}; \quad \alpha_2 = 0.1250 \frac{p}{a^6 b^2}; \quad \alpha_3 = 0.01826 \frac{p}{a^6 b^2}.$$

* Similar results were obtained by Prof. C. E. Inglis in the case of a sinusoidal distribution of tractions on the edges $x = \pm a$. See paper, "Two-dimensional Stresses in Rectangular Plates," read before Section G of the British Association at Edinburgh, September 1921.

The corresponding value of stresses on the cross-section ($x=0$) will be

$\frac{y}{a}$	=	0	0.2	0.4	0.6	0.8	1.0
$(X_x)_{x=0}$	=	0.690 p	0.684 p	0.669 p	0.653 p	0.649 p	0.675 p

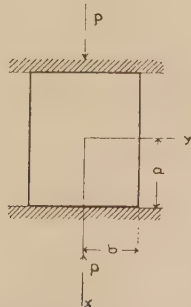
This distribution is represented on fig. 2 by a dotted line. We see that the deviation from the average stress (equal to $\frac{2}{3}p$) is now within the limits of accuracy of our approximate solution.

In the case of other symmetrical distributions of tractions over the edges $x=\pm a$ we have only to change the form of the function ϕ_0 in the expression (9): then the right-hand expressions only, in equations (10), have to be changed.

It is well known that if a thin plate, initially flat, be bent by couples and forces applied at its edges, the equation satisfied by the principal curvatures of its surface is mathematically identical with the equation (1)*. Hence, any solution of the two-dimensional problem can be used for investigating the bending of a thin plate with appropriate conditions at the boundary. The problem explained above corresponds to the bending of a rectangular plate, the deflexions (w) of which satisfy the following conditions at the boundary:

$$\left(\frac{dw}{dy}\right)_{y=\pm b} = \pm \frac{2}{3}pb; \quad \left(\frac{d^2w}{dy^2}\right)_{x=\pm a} = p\left(1 - \frac{y^2}{b^2}\right); \quad \left(\frac{dw}{dx}\right)_{x=\pm a} = 0.$$

Fig. 3.



§ 3. As a second example, we will consider the problem of a block between two perfectly rigid bearing-plates (fig. 3). We suppose that horizontal displacements on the contact surfaces are prevented by friction. The conditions on the free boundary of the block will be satisfied if

* This identity can be used in experimental solutions of two-dimensional problems. See K. Wieghardt, 'Mitteilungen u. Forschungsarbeiten,' H. 49 (1908).

1100 Prof. S. Timoschenko on the Approximate Solution
we take the stress-function in the form

$$\phi = -\frac{py^2}{2} + (y^2 - b^2)^2 (\alpha_1 + \alpha_2 x^2 + \alpha_3 x^4 + \alpha_4 x^2 y^2 + \dots), \quad (11)$$

where p denotes the value of the average compressive stress.

Limiting ourselves to the four terms in brackets, we obtain the following expressions for the stress-components:—

$$X_x = -p + (12y^2 - 4b^2)(\alpha_1 + \alpha_2 x^2 + \alpha_3 x^4) \\ + \alpha_4 x^2 (30y^4 - 24y^2 b^2 + 2b^4),$$

$$Y_y = 2(y^2 - b^2)^2 (\alpha_2 + 6\alpha_3 x^2 + \alpha_4 y^2),$$

$$X_y = -8y(y^2 - b^2)(\alpha_2 x + 2\alpha_3 x^3) - 4\alpha_4 x(3y^5 - 4y^3 b^2 + yb^4).$$

We see that, with the assumption made in regard to the displacements at the contact surfaces, the right side of (5) will be equal to zero. The stress-distribution in the case considered depends on the elastic constants, and we must take the equations for calculating the coefficients $\alpha_1, \alpha_2, \dots$ in the form (8). After performing the integrations, we obtain these equations in the following form:—

$$\alpha_1 + \alpha_2 a^2 \left(\frac{1}{3} + \sigma \frac{4}{21} \frac{b^2}{a^2} \right) + \alpha_3 a^4 \left(\frac{1}{5} + \sigma \frac{8}{21} \frac{b^2}{a^2} \right) + \alpha_4 a^4 \frac{1}{21} \frac{b^2}{a^2} = 0,$$

$$\alpha_1 \left(4 + \sigma \frac{16}{7} \frac{b^2}{a^2} \right) + \alpha_2 a^2 \left[\frac{12}{5} + \frac{32}{21} \frac{b^4}{a^4} + (2 + 3\sigma) \frac{32}{21} \frac{b^2}{a^2} \right] \\ + \alpha_3 a^4 \left[\frac{12}{7} + \frac{64}{21} \frac{b^4}{a^4} + \sigma \frac{112}{35} \frac{b^2}{a^2} + (1 + \sigma) \frac{128}{35} \frac{b^2}{a^2} \right] \\ + \alpha_4 a^4 \left(\frac{12}{35} \frac{b^2}{a^2} + \frac{32}{231} \frac{b^6}{a^6} \right) = -\frac{\sigma p}{a^2},$$

$$\alpha_1 \left(\frac{6}{5} + \sigma \frac{16}{7} \frac{b^2}{a^2} \right) + \alpha_2 a^2 \left[\frac{6}{7} + \frac{32}{21} \frac{b^4}{a^4} + \sigma \frac{8}{5} \frac{b^2}{a^2} + (1 + \sigma) \frac{64}{35} \frac{b^2}{a^2} \right] \\ + \alpha_3 a^4 \left[\frac{3}{2} + \frac{576}{105} \frac{b^4}{a^4} + \sigma \frac{96}{49} \frac{b^2}{a^2} + 2(1 + \sigma) \frac{64}{49} \frac{b^2}{a^2} \right] \\ + \alpha_4 a^4 \left[\frac{6}{49} \frac{b^2}{a^2} + \frac{32}{231} \frac{b^6}{a^6} \right] = -\frac{\sigma p}{a^2},$$

$$\alpha_1 \cdot 4 + \alpha_2 a^2 \left(\frac{12}{5} + \frac{32}{33} \frac{b^4}{a^4} \right) + \alpha_3 a^4 \left(\frac{12}{7} + \frac{64}{33} \frac{b^4}{a^4} \right) + \alpha_4 a^4 \left[\frac{36}{5} \frac{b^2}{a^2} \right. \\ \left. + \frac{32}{143} \frac{b^6}{a^6} + (2 + 3\sigma) \frac{32}{33} \frac{b^4}{a^4} \right] = -\frac{\sigma p}{a^2}.$$

We will make the calculations in the case of a square block.

Putting $a=b$, $\sigma=\frac{1}{4}$, we get the following values of coefficients:

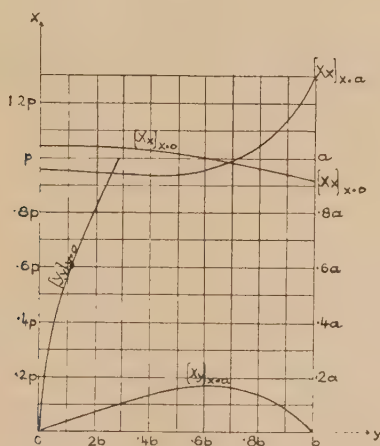
$$\alpha_1 = 0.01016 \frac{p}{a^2}; \quad \alpha_2 = -0.00629 \frac{p}{a^4};$$

$$\alpha_3 = -0.02336 \frac{p}{a^6}; \quad \alpha_4 = -0.01825 \frac{p}{a^6}.$$

The corresponding values of the stress-components on the contact surfaces and on the cross-sections ($x=0$) and ($y=0$) are given in the table below:

$\frac{x}{a} = \frac{y}{b} =$	0	0.2	0.4	0.6	0.8	0.9	1.0
$X_x)_{x=\pm a} \dots$	-0.959 p	-0.951 p	-0.940 p	-0.956 p	-1.052 p	-1.152 p	-1.302 p
$X_x)_{x=0} \dots$	-1.041 p	-1.033 p	-1.021 p	-0.997 p	-0.963 p	-0.942 p	-0.919 p
$X_y)_{x=\pm a} \dots$	0	0.069 p	0.130 p	0.165 p	0.141 p	0.090 p	0
$X_y)_{y=0} \dots$	-0.013 p	-0.024 p	-0.057 p	-0.113 p	-0.192 p	-0.240 p	-0.292 p

Fig. 4.

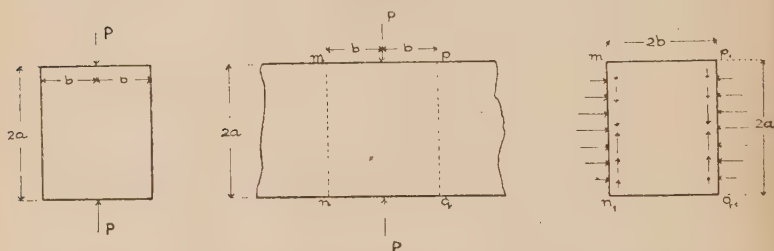


This stress-distribution is represented in fig. 4. We see that the compressive stresses are not distributed uniformly on the contact surfaces. There is some decrease of stress in

the middle, and a considerable increase near* the edges ($y = \pm b$). There are also tangential stresses acting on the contact surfaces. The maximum value of these is about 16 per cent. of the average compressive stress (fig. 4).

The block problem in the case of compression with two concentrated forces (fig. 5a) can be solved by combining the known solution for an infinite strip† (fig. 5b) with that for a block (fig. 5c), on the edges m_1n_1 and p_1q_1 of which

Fig. 5.



tractions corresponding to the stresses on the cross-sections mn and pq of the strip act. The last problem can be solved in the manner explained in the first example above.

§ 4. The same method can be used in order to estimate the additional deflexion of a beam due to shearing. The boundary conditions will be satisfied if we take the stress-function in the following form :

$$\phi = \alpha_1 \left[xy + \frac{1}{3c^2} (l-x)y^3 \right] + \alpha_2 e^{-\beta x} (y^2 - c^2)^2 y. \quad (13)$$

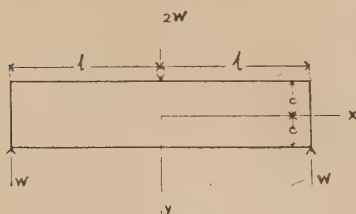
The corresponding stress-components will be

$$\left. \begin{aligned} X_x &= \frac{2\alpha_1}{c^2} (l-x)y + \alpha_2 e^{-\beta x} (20y^3 - 12c^2y), \\ Y_y &= \alpha_2 \beta^2 e^{-\beta x} (y^2 - c^2)^2 y, \\ X_y &= -\frac{\alpha_1}{c^2} (c^2 - y^2) + \alpha_2 \beta e^{-\beta x} (5y^4 - 6y^2c^2 + c^4). \end{aligned} \right\} \quad (14)$$

* In order to calculate these stresses of local character with more accuracy, it would be necessary to increase the number of terms in expression (11), which will entail considerable more calculating work. The calculations made on the compression of the block suggest that the maximum compressive stress obtained above is somewhat exaggerated.

† See Pigeaud, *Comptes Rendus* (Paris, 1915), p. 673.

Fig. 6.



Substituting these expressions in (2), and taking by calculation

$$\int_0^l e^{-\beta x} dx = \frac{1}{\beta},$$

we obtain

$$\begin{aligned} V = \frac{1}{E} \left\{ \frac{8}{9} \alpha_1^2 \frac{l^3}{c} + \frac{64}{7} \alpha_2^2 \frac{c^7}{\beta} + \frac{128}{11 \cdot 9 \cdot 7 \cdot 5} \alpha_2^2 c^{11} \beta^3 \right. \\ \left. - \frac{64}{105} \sigma \alpha_1 \alpha_2 \cdot c^5 (\beta l - 1) + \frac{256}{9 \cdot 7 \cdot 5} \sigma \alpha_2^2 c^9 \beta \right. \\ \left. + 2(1 + \sigma) \left(\frac{16}{15} \alpha_1^2 c l + \frac{128}{9 \cdot 7 \cdot 5} \alpha_2^2 c^9 \beta - \frac{64}{105} \alpha_1 \alpha_2 c^5 \right) \right\}. \end{aligned} \quad (15)$$

The terms involving α_1 , in the expressions (14), correspond to the usual formula for the bending of a narrow rectangular beam. The quantities α_2 and β , corresponding to the "local irregularity" of stress-distribution near the cross-section ($x=0$), we will choose in such a way as to make V a minimum. We thus obtain the equation:

$$\alpha_2 = m \frac{\alpha_1 l}{c^4}, \quad (16)$$

where

$$m = \frac{64}{105} \sigma (\beta c)^2 \cdot \frac{1}{\frac{384}{11 \cdot 9 \cdot 7 \cdot 5} (\beta c)^4 + (1 + 2\sigma) \frac{256}{9 \cdot 7 \cdot 5} (\beta c)^2 - \frac{64}{7}},$$

and

$$\begin{aligned} & \frac{(2 + \sigma)}{2\sigma} \left[\frac{384}{11 \cdot 9 \cdot 7 \cdot 5} (\beta c)^4 + \frac{256}{9 \cdot 7 \cdot 5} (1 + 2\sigma) (\beta c)^2 - \frac{64}{7} \right] \\ & = \frac{l}{2c} \left[-\frac{128}{11 \cdot 9 \cdot 7 \cdot 5} (\beta c)^5 + (1 + 2\sigma) \frac{256}{9 \cdot 7 \cdot 5} (\beta c)^3 + \frac{192}{7} (\beta c) \right]. \end{aligned} \quad (17)$$

From (17) we obtain* values of $\frac{l}{2c}$ corresponding to different values of βc (see table below). With these values of βc we can now calculate, from (16), the values of α_2 . Substituting βc and α_2 in the expression (15) and equalizing it to the work W_f done by the load $2W$, we obtain the deflexion f from the equation

$$f = \frac{Wl^3}{2Ec^3} + \frac{6}{5} \frac{Wl}{2\mu c} - n \frac{Wl}{2\mu c} - q \frac{Wl^2}{2Ec^2}, \quad (18)$$

where

$$\mu = \frac{E}{2(1+\sigma)}; \quad n = \frac{12}{35} \frac{2+\sigma}{1+\sigma} m;$$

$$q = \frac{9}{8} \frac{m}{\beta c} \left\{ \sigma \frac{64}{105} (\beta c)^2 - m \left[\frac{128}{11.9.7.5} (\beta c)^4 + \frac{2(1+2\sigma)}{9.7.5} 128 (\beta c)^2 + \frac{64}{7} \right] \right\}.$$

Some values of n and q are given in the table below :

βc	=	6	6.5	7	7.07
$\frac{l}{2c}$	=	4.39	8.74	72.0	∞
m	=	0.0363	0.0317	0.0279	
n	=	0.022	0.019	0.017	
q	=	0.019	0.020	0.020	

We see that we may write $n=q=0.02$, in (18), without serious inaccuracy. Then the deflexion may be represented as follows :

$$f = \frac{Wl^3}{2Ec^3} \left(1 + 2.95 \frac{c^2}{l^2} - 0.02 \frac{c}{l} \right). \quad . \quad . \quad . \quad (19)$$

In this manner we can approximately estimate the effect of "local irregularity" of stress-distribution due to the condition that the middle cross-section remains plane

March 7th, 1922.

* In the calculations we have taken $\sigma = \frac{1}{4}$.

CVII. *Diffraction Image of Two Close Luminous Lines of Finite Width.* By B. E. MOURASHKINSKY, Optical Laboratory of the Central Chamber of Weights and Measures, Petrograd *.

THE illumination at a point P in a focal plane of a geometrically corrected object-glass with a circular aperture due to an infinite narrow line source is expressed by

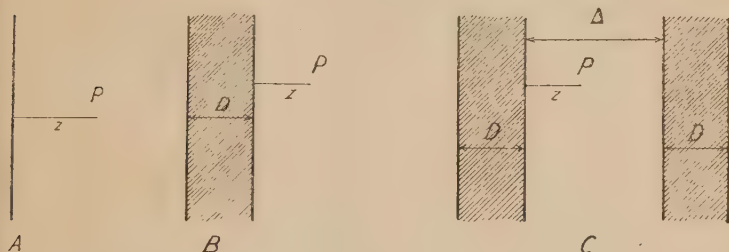
$$I(z) = 3L(z), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$z = \frac{2\pi R}{\lambda f} r,$$

r being the linear distance of a point P from the geometrical image of the line source (fig. 1 A).

Fig. 1.



$L(z)$ is the function of Lord Rayleigh :

$$L(z) = \frac{\pi}{2} \frac{H_1(2z)}{(2z)^3}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $H_1(2z)$ is H. Struve's † function :

$$\begin{aligned} H_1(z) &= \frac{2z^2}{\pi} \int_0^{\frac{\pi}{2}} \sin(z \sin \theta) \cos \theta d\theta \\ &= \frac{2}{\pi^2} \left[\frac{z^3}{1^2 \cdot 3} - \frac{z^5}{1^2 \cdot 3^2 \cdot 5} + \frac{z^7}{1^2 \cdot 2^2 \cdot 5^2 \cdot 7} - \dots \right]. \quad (3) \end{aligned}$$

* Communicated by the Author.

† Struve, "Über die Einfluss der Diffraction an Fernrohren auf Lichtscheiben," *Mémoires de l'Académie des Sciences de St. Pétersbourg*, xxx. no. 8 (1882), and *Ann. der Phys.* xvii. p. 1008 (1882).

The function $L(z)$ * by means of (3) may be represented by the series

$$L(z) = \frac{1}{1^2 \cdot 3} - \frac{(2z)^2}{1^2 \cdot 3^2 \cdot 5} + \frac{(2z)^4}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7} - \dots \quad (4)$$

The illumination at a point P given by (1) is expressed in terms of illumination at the geometrical image of a luminous line. We have no complete tables for the distribution of illumination in a diffraction image of a fine line due to a circular aperture except those by André †, calculated by another method not clearly explained in his paper.

As to the limit of resolution of two infinite narrow luminous lines, Lord Rayleigh gives the value 3.14 as the least distance between two lines in order they may be just resolved. The percentage difference between the illumination at its minimum (at the centre of diffraction image) and that at the geometrical images of each line is 4.5. The displacement of maxima (see below) was not considered in Lord Rayleigh's paper.

The illumination at the point P due to a luminous line of finite width may be found in the following manner:—

The expression (1) is to be integrated between the limits z and $z + D$, where

$$D = \frac{2\pi R}{\lambda f} d,$$

d being the linear width of the geometrical image of the line in a focal plane. The z represents now (fig. 1 B) the distance of a point P from the image of the nearer edge of the line. If the point P is outside of the geometrical image of the line, z will be positive; and if P is inside of it, z will be negative. Thus the position of the axis of the line will be noted as

$$z = -\frac{D}{2}.$$

The illumination at a point P is written

$$I(z) = 3 C \int_z^{z+D} L(z) dz. \quad \dots \quad (5)$$

The constant C depends on the choice of units. If we express the illumination at P due to a line of finite width in terms of illumination at the same point due to the luminous

* Lord Rayleigh, "Wave Theory," Encyclopædia Britannica.

† Ch. André, 'Etude de la Diffraction dans les Instruments d'optique,' 1876.

plane infinite in all directions and having the same brightness as the line object, the constant C will be *

$$C = \frac{8}{3\pi^2} \quad \dots \quad (6)$$

and

$$I(z) = \frac{8}{\pi^2} \int_z^{z+D} L(z) dz \quad \dots \quad (7)$$

This expression may be written

$$I(z) = \frac{8}{\pi^2} \int_z^\infty L(z) dz - \frac{8}{\pi^2} \int_{z+D}^\infty L(z) dz \quad \dots \quad (8)$$

Each of the integrals (8) represents the illumination at a point P due to a semi-infinite luminous plane with straight edge, a point P being at the distances z or $z+D$ from the geometrical image of the edge. For the integrals of this type we have the tables in Struve's paper mentioned above.

At the axis of the geometrical image of a line of finite width we have illumination

$$I\left(-\frac{D}{2}\right) = 1 - 2 \cdot \frac{8}{\pi^2} \int_{\frac{D}{2}}^\infty L(z) dz \quad \dots \quad (9)$$

At the geometrical image of the edges of a line the illumination will be

$$I(0) = \frac{1}{2} - \frac{8}{\pi^2} \int_D^\infty L(z) dz \quad \dots \quad (10)$$

The curve representing (8) has a fluctuating character, with maxima and minima. As D increases, the maxima become less and less marked.

The position of maxima and minima and the values of D at which there are no maxima and minima are very difficult to find analytically; they are obtained by a graphical and computative method for 1st minimum, 1st maximum, 2nd minimum, and 2nd maximum. These values are given in Table I.

* B. E. Mourashkinsky, "Diffraction Image of Two Close Luminous Planes," *Phil. Mag.* xlv. p. 802 (1923).

TABLE I.

z	$D = 0.1$	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\frac{D}{2}$	0.270									
0	$\begin{cases} a. 0.270 \\ b. 1.0000 \end{cases}$	0.539 1.0000	1.063 0.9888	1.576 0.9789	2.045 0.9542	2.479 0.9376	2.868 0.9122	3.207 0.8844	3.496 0.8548	3.735 0.8234	3.927 0.7921
1st min. ...	$\begin{cases} a. 0.008 \\ b. 0.296 \\ c. +3.6 \end{cases}$	0.016 0.296 +3.5	0.034 0.315 +3.4	0.053 0.329 +3.4	0.074 0.348 +3.3	0.096 0.364 +3.3	0.118 0.375 +3.2	0.143 0.394 +3.2	0.168 0.410 +3.2	0.192 0.423 +3.2	0.212 0.427 +3.2
1st max. ...	$\begin{cases} a. 0.012 \\ b. 0.484 \\ c. +4.6 \end{cases}$	0.025 0.463 +4.5	0.050 0.463 +4.4	0.074 0.460 +4.4	0.096 0.450 +4.2	0.118 0.447 +4.0	0.139 0.442 +4.0	0.159 0.438 +4.0	0.178 0.435 +3.8	0.195 0.429 +3.6	0.212 0.427 +3.5
2nd min.	$\begin{cases} a. 0.003 \\ b. 0.111 \\ c. +6.6 \end{cases}$	0.006 0.111 +6.6	0.012 0.111 +6.6	0.018 0.112 +6.5	0.024 0.113 +6.4	0.031 0.117 +6.4	0.039 0.124 +6.3	0.047 0.129 +6.3	0.055 0.134 +6.3	0.063 0.139 +6.3	0.071 0.143 +6.3
2nd max.	$\begin{cases} a. 0.004 \\ b. 0.150 \\ c. +7.7 \end{cases}$	0.008 0.150 +7.7	0.016 0.150 +7.7	0.024 0.150 +7.6	0.032 0.150 +7.5	0.040 0.150 +7.4	0.047 0.149 +7.3	0.054 0.149 +7.2	0.060 0.147 +7.1	0.066 0.145 +6.9	0.072 0.145 +6.7
z	$D = 3.0$	4.0	5.0	6.0	7.0	8.0	9.0	10.0			
$-\frac{D}{2}$	6716	7854	8740	8856	8944	9056	9180			
0 ... $\begin{cases} a. \\ b. \end{cases}$	$\begin{cases} a. 4370 \\ b. 6507 \end{cases}$	4472 5694	4590 5419	4672 5346	4707 5315	4744 5304	4778 5276	4797 5225			

In Table I. the data noted *a* are the values of illumination calculated by means of (8), those noted *b* the same values referred to the illumination at the axis, and the data noted *c* the positions of maxima or minima (the distances from the edge of the line). The data of this table, as those of the following tables of illumination, are correct to 2-4 units of fourth decimal.

It is seen that even for the width of the luminous line $D=2.0$ two neighbouring maxima and minima differ one from the other only by a very small amount. For values of $D>2.0$ the curves representing the distribution of illumination have no sensible maxima or minima. The illumination at the geometrical image of the edges referred to that at the axis decreases rapidly with increasing D for small values of D and very slowly for large ones, asymptotically approaching the value 0.5 for the semi-infinite plane.

If we have two lines of finite width D at the distance Δ between their inner edges, the illumination at any point P in the focal plane is calculated from the expression :

$$I(z) = \frac{8}{\pi^2} \int_z^\infty L(z) dz - \frac{8}{\pi^2} \int_{z+D}^\infty L(z) dz + \frac{8}{\pi^2} \int_{\Delta-z}^\infty L(z) dz - \frac{8}{\pi^2} \int_{\Delta-z+D}^\infty L(z) dz. \quad (11)$$

The values of the integrals (11) are to be taken directly from Struve's tables. At the axis of the image of each line the illumination will be

$$I\left(-\frac{D}{2}\right) = 1 - 2 \cdot \frac{8}{\pi^2} \int_{\frac{D}{2}}^\infty L(z) dz + \frac{8}{\pi^2} \int_{\Delta+\frac{D}{2}}^\infty L(z) dz - \frac{8}{\pi^2} \int_{\Delta+\frac{3}{2}D}^\infty L(z) dz. \quad (12)$$

At the centre of the diffraction image of two luminous lines the illumination will be

$$I\left(\frac{\Delta}{2}\right) = 2 \left[\frac{8}{\pi^2} \int_{\frac{\Delta}{2}}^\infty L(z) dz - \frac{8}{\pi^2} \int_{\frac{\Delta}{2}+D}^\infty L(z) dz \right]. \quad (13)$$

In the expressions (11), (12), and (13)

$$\Delta = \frac{2\pi R}{\lambda f} b,$$

b being the linear distance (in a focal plane) between the geometrical images of the inner edges (fig. 1 c).

It is evident that two luminous lines of width D are resolved by the same object-glass at different distances Δ between their inner edges for different values of D . The resolution of two such objects depends on the distribution of illumination in the diffraction image or on the contrast between the illumination at its minimum and that at maximum. This contrast will vary with D and Δ .

The angular width of a line and that of the interspace between them are expressed in D and Δ and *vice versa* by means of the following relations :

$$\left. \begin{aligned} D &= \frac{2\pi R}{\lambda} \cdot \tan D'', \\ \Delta &= \frac{2\pi R}{\lambda} \tan \Delta''. \end{aligned} \right\} \dots \dots (14)$$

For given angular values of D'' and Δ'' different object-glasses give different values of D and Δ .

To investigate the diffraction conditions of the resolution of such pairs of close luminous lines of finite width, the values of illumination at the axes of the lines and at the centre of the image were calculated by means of (12) and (13) for different values of D and Δ from 0.2 to 10.0. Thus we can express the illumination at its minimum (at the centre of image) in terms of illumination at the axes of each line for each D and different Δ . Plotting these on millimetre paper (the Δ as abscissæ and the illuminations as ordinates), we have the percentage illumination curves. Extrapolating them graphically, we obtain the values of Δ for which the illumination at its minimum is equal to that at the axes of two lines. The distances thus obtained might be called the critical distances, unless there were not the displacements of maxima for the small values of Δ . The maxima of illumination for small values of Δ do not coincide with the positions of the geometrical images of the axes of two lines, but are displaced symmetrically towards the centre of the image. To find the values of true maxima and values of their displacements with respect to the axes, the illumination was calculated by means of (11) for the whole illumination curve, especially for the points near the axes, and the values of the maxima and their displacements were found graphically. The values of the displacements of the maxima are given in Table II. So as to insert all the data in one table, the distances between two lines are given not in Δ , but in distances between the axes $\Delta + D$. The

TABLE II.

[illegible]

displacements are expressed in the same units as D or Δ ; thus for $D=0.2$ and $\Delta+D=2.9$ (*i. e.* $\Delta=2.7$) the maxima are displaced from the axes by 0.76 in the same units towards the centre of the diffraction image.

As we see from Table II., the values of displacements of maxima decrease with increasing $\Delta+D$ and generally increase with increasing D for a given $\Delta+D$. As these values are obtained graphically, they are only correct to 3 4 units of the second decimal place, and for this reason the last displacements are not given for less than 0.04 . The distances $\Delta+D$ for which the displacements of maxima are less than 0.04 may be considered as having no displacements. From the same table we see that for the small values of D and $\Delta+D$ the maxima may occur even outside of the geometrical images of lines. For example, for $D=0.4$ and $\Delta+D=3.0$ (*i. e.* $\Delta=2.6$) the displacement of maxima is 0.56 (in the same units); thus they are 0.36 outside of the geometrical images of the inner edges of the lines.

The complete tables of distribution of illumination in diffraction images of two luminous lines for different D and Δ are too long to be given here in full. The values of illumination at the most interesting points of the illumination curves for a number of D are given here as a short abstract of the computed tables. In Table III. M_0 denotes the illumination at the axis of each line, M the maxima of illumination, m the illumination at the central minimum between geometrical images of lines. In the last column are given the percentage ratios of illumination at its minimum to that at the axes or at maxima. We see that values under M_0 have a fluctuating character depending on the fluctuations of the illumination curves of each line taken separately. So that, for example, the values of M_0 for $D=0.2$ have a minimum for $\Delta=3.4$ and a maximum for $\Delta=4.4$; but as is seen from Table I., the luminous line of width $D=0.2$ has its first minimum at 3.5 and first maximum at 4.5 , *i. e.* just on the axis of the other line of width $D=0.2$. Analogously for $D=0.8$ the values of M_0 have a minimum for $\Delta=2.9$ and maximum for $\Delta=3.8$. From Table I. we have that the line of width $D=0.8$ has its first minimum at 3.3 and first maximum at 4.2 , so that $3.3 - \frac{D}{2} = 3.3 - 0.4 = 2.9$ and $4.2 - \frac{D}{2} = 3.8$. For the values of $D=2.0$ and larger we have not these fluctuations of M_0 .

TABLE III.

 $D = 0.2.$

$\Delta.$	$M_0.$	$M.$	$m.$	$\frac{m}{M_0} \cdot 100.$	$\frac{m}{M} \cdot 100.$
2.7	·0580	·0605	·0602	103.8	99.5
2.8	·0573	·0588	·0578	100.9	98.3
3.0	·0564	·0569	·0526	93.3	92.4
3.2	·0559	·0560	·0478	85.6	82.4
3.4	·0556		·0432		77.7
3.6	·0558		·0384		68.8
3.8	·0561		·0340		60.6
4.0	·0562		·0300		53.4
4.2	·0563		·0262		46.5
4.4	·0566		·0226		39.9
4.6	·0564		·0194		34.4
4.8	·0562		·0164		29.2
5.0	·0561		·0138		24.6

 $D = 0.4.$

$\Delta.$	$M_0.$	$M.$	$m.$	$\frac{m}{M_0} \cdot 100.$	$\frac{m}{M} \cdot 100.$
2.5	·1159	·1209	·1206	104.1	99.8
2.6	·1146	·1174	·1154	100.7	98.3
2.8	·1127	·1134	·1056	93.7	93.1
3.0	·1116	·1118	·0958	85.8	85.7
3.2	·1112		·0862		77.5
3.4	·1114		·0772		69.3
3.6	·1119		·0682		60.9
3.8	·1122		·0602		53.7
4.0	·1125		·0526		46.8
4.2	·1128		·0456		40.4
4.4	·1127		·0390		34.5
4.6	·1123		·0332		29.6
4.8	·1119		·0280		25.0
5.0	·1114		·0234		21.0

 $D = 0.6.$

$\Delta.$	$M_0.$	$M.$	$m.$	$\frac{m}{M_0} \cdot 100.$	$\frac{m}{M} \cdot 100.$
2.3	·1736	·1813	·1808	104.2	99.7
2.4	·1715	·1760	·1734	101.1	98.5
2.6	·1686	·1698	·1586	94.1	93.4
2.8	·1671		·1440		86.2
3.0	·1663		·1298		78.1
3.2	·1665		·1162		69.8
3.4	·1671		·1034		61.9
3.6	·1676		·0910		54.3
3.8	·1681		·0796		47.4
4.0	·1683		·0690		41.0
4.2	·1683		·0594		35.3
4.4	·1677		·0506		30.2
4.6	·1671		·0428		25.6
4.8	·1663		·0360		21.6
5.0	·1655		·0300		18.0

Table III. (*continued*).

D = 0.8.					
Δ .	M_0 .	M.	m .	$\frac{m}{M_0} \cdot 100$.	$\frac{m}{M} \cdot 100$.
2.1	2306	2416	2414	104.7	99.9
2.2	2279	2343	2316	101.6	98.8
2.4	2238	2258	2118	94.6	93.8
2.6	2215	2219	1926	87.0	86.8
2.8	2206		1740		78.9
3.0	2207		1560		70.7
3.2	2212		1388		62.7
3.4	2220		1228		55.3
3.6	2226		1074		48.3
3.8	2228		0934		41.9
4.0	2227		0806		36.2
4.2	2222		0690		31.1
4.4	2213		0586		26.5
4.6	2203		0494		22.4
4.8	2193		0416		19.0
5.0	2182		0348		15.9

D = 1.0.					
Δ .	M_0 .	M.	m .	$\frac{m}{M_0} \cdot 100$.	$\frac{m}{M} \cdot 100$.
1.9	2873	3019	3016	105.0	99.9
2.0	2837	2924	2806	102.1	99.0
2.2	2784	2813	2656	95.4	94.4
2.4	2754	2761	2418	87.8	87.6
2.6	2742		2188		79.8
2.8	2740		1966		71.8
3.0	2744		1754		63.9
3.2	2754		1552		56.4
3.4	2760		1366		49.5
3.6	2762		1190		43.1
3.8	2760		1030		37.3
4.0	2755		0886		32.2
4.2	2747		0756		27.5
4.4	2733		0642		23.5
4.6	2719		0542		19.9
4.8	2708		0458		16.9
5.0	2692		0386		14.3

Table III. (continued).

D = 2.0.					
Δ .	M_0 .	M.	m .	$\frac{m}{M_0} \cdot 100$.	$\frac{m}{M} \cdot 100$.
1.0	·5454	·5639	·5612	103.3	99.5
1.2	·5375	·5503	·5400	100.5	98.1
1.4	·5284	·5333	·4982	94.3	93.4
1.6	·5227	·5240	·4570	87.4	87.2
1.8	·5194		·4168		80.2
2.0	·5178		·3782		73.0
2.2	·5171		·3412		66.0
2.4	·5169		·3060		59.2
2.6	·5159		·2730		52.8
2.8	·5166		·2424		46.9
3.0	·5158		·2140		41.5
3.2	·5145		·1880		36.5
3.4	·5131		·1646		32.1
3.6	·5112		·1436		28.1
3.8	·5093		·1250		24.5
4.0	·5075		·1090		21.5

D = 3.0.

Δ .	M_0 .	M.	m .	$\frac{m}{M_0} \cdot 100$.	$\frac{m}{M} \cdot 100$.
0.3	·7545	·7976	·7974	105.7	100.0
0.4	·7456	·7757	·7718	103.5	99.5
0.6	·7310	·7451	·7208	98.6	96.7
0.8	·7199	·7266	·6706	93.2	92.3
1.0	·7120	·7150	·6212	87.2	86.9
1.2	·7066	·7078	·5728	81.1	80.9
1.4	·7030	·7033	·5262	74.9	74.8
1.6	·7007		·4816		68.7
1.8	·6992		·4388		62.8
2.0	·6980		·3986		57.1
2.2	·6970		·3608		51.8
2.4	·6958		·3252		46.7
2.6	·6943		·2922		42.0
2.8	·6928		·2618		37.8
3.0	·6901		·2362		34.2
3.2	·6894		·2090		30.3
3.4	·6878		·1866		27.1
3.6	·6863		·1662		24.2
3.8	·6851		·1482		21.6
4.0	·6840		·1326		19.4

Plotting the data of the last columns of all the Tables III. for each D on millimetre paper, we obtain the percentage illumination curves. Extrapolating them graphically, we have the distances for which the illumination at central minimum

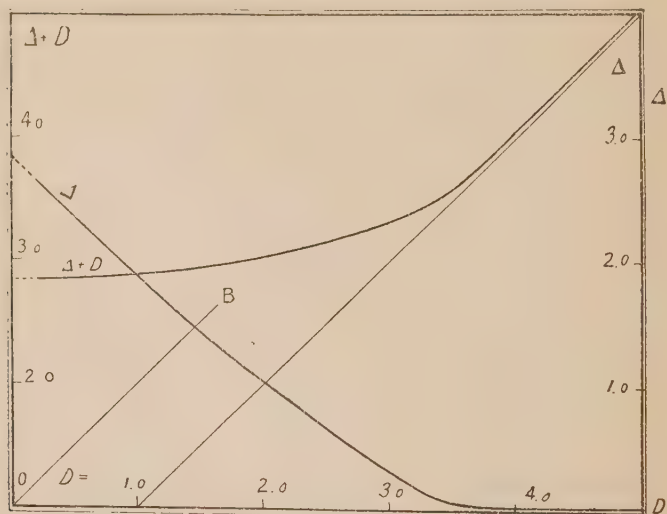
is equal to that at maximum (generally not at the axes), for different values of D from 0.2 to 10.0. These distances are the *critical distances* and are considered as *absolute limits of resolution* of these lines. The critical distances thus obtained, expressed in Δ or in $\Delta + D$, are given in Table IV. and fig. 2.

TABLE IV.

D.	$\Delta_{cr.}$	$\Delta + D_{cr.}$	D.	$\Delta_{cr.}$	$\Delta + D_{cr.}$	D.	$\Delta_{cr.}$	$\Delta + D_{cr.}$
0.2	2.65	2.85	2.2	.90	3.10	4.2	.04	4.24
0.4	2.55	2.85	2.4	.75	3.15	4.4	.04	4.44
0.6	2.26	2.86	2.6	.60	3.20	4.6	.04	4.64
0.8	2.07	2.87	2.8	.45	3.25	4.8	.04	4.84
1.0	1.88	2.88	3.0	.30	3.30	5.0	.04	5.04
1.2	1.70	2.90	3.2	.20	3.40	6.0	<.04	<6.04
1.4	1.52	2.92	3.4	.10	3.50	7.0	<.04	<7.04
1.6	1.36	2.96	3.6	.06	3.66	8.0	<.04	<8.04
1.8	1.20	3.00	3.8	.05	3.85	9.0	<.04	<9.04
2.0	1.05	3.05	4.0	.05	4.05	10.0	<.04	<10.04

Fig. 2.

4.0



We see that for larger values of D the critical distances are less, so that the broader lines are resolved at less distances between edges. For small values of D the critical distances decrease rapidly; for the large ones they decrease very slowly, approaching asymptotically zero for two semi-infinite luminous planes.

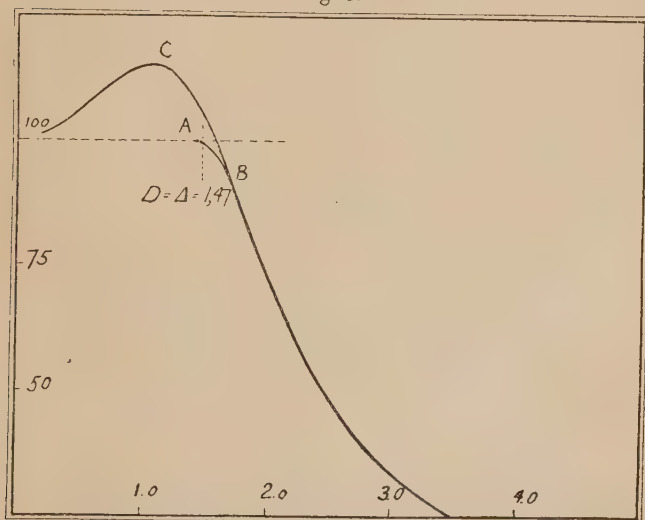
If we have two luminous lines of variable width D with distance Δ between the inner edges, Δ being always equal to

the width of each line, the critical distance for this case may be found graphically by means of the curve Δ_{cr} of fig. 2. The intersection of the line OB ($D=\Delta$) with the curve Δ_{cr} gives the critical distance for $D=\Delta$; it is equal to 1.47, or $D+\Delta=2.94$. (The critical distance for two luminous points of equal brightness is equal to 3.0*.)

TABLE V.

$\Delta=D.$	$\frac{m}{M_0} \cdot 100.$	$\frac{m}{M} \cdot 100.$	$\Delta=D.$	$\frac{m}{M_0} \cdot 100.$
1.6	96.6	98.1	4.2	16.7
1.8	85.4	85.5	4.4	14.9
2.0	73.0		4.6	13.5
2.2	61.7		4.8	12.3
2.4	52.5		5.0	11.4
2.6	44.9			
2.8	38.8		6.0	9.2
3.0	34.2		7.0	8.5
3.2	29.8		8.0	7.8
3.4	26.4		9.0	7.0
3.6	23.5		10.0	6.2
3.8	20.9			
4.0	18.7			

Fig. 3.



As the case $D=\Delta$ may be of some practical interest in treating resolving powers of an object-glass, the values of illumination at the centre of image with reference to that at maximum are given in Table V. and fig. 3.

* B. E. Mourashkinsky, "Diffraction Pattern in a case of two very close Point-Light Sources," *Phil. Mag.* xlv. p. 29 (1923).

The intersection of the curve AB with the line 100 (fig. 3) gives once more the critical distance for this case. It is equal, as said, to 1.47 if the displacements of maxima, as it is necessary, are taken into account. For comparison sake, the curve of percentage illumination at the centre of image with reference to that at axes is given in fig. 3 (the curve BC). In this case the distance at which the illumination at the centre is equal to that at the axis is equal to 1.58. For the values $\Delta=D$ greater than 1.84, both the curves coincide.

Extrapolating the curve Δ_{cr} in fig. 2, we obtain the critical distance for $D=0$ (two infinite narrow luminous lines); it is equal to 2.85. Neglecting the displacements of maxima and extrapolating the curve of distances, for which the illumination at the centre is equal to that at the axes, we have that for $D=0$ the corresponding distance is 3.03.

In treating the problem of resolution of two close luminous lines of finite width by an object-glass, it is necessary to know the distribution of illumination along a line perpendicular to geometrical images of the edges of each line. But besides that it is necessary to know the contrast sensibility of the eye. The latter is known only for a case of two adjacent fields for different values of brightness of the fields; but for the diffraction image of two luminous objects, as two luminous lines, the case of adjacent fields cannot be applied. There are, as far as we know, no experimental determinations of contrast sensibility of the eye for a distribution of illumination having a minimum with rapid or slow increase. *A priori*, it might be said that for different values of brightness of the sources the contrast sensibility of the eye will be different.

Assuming the contrast sensibility of the eye to be equal to 5, 10, 15, 20, or 25 per cent., the limit distances Δ between the edges of two lines of different width D could be obtained from the curves representing the last columns of Table IV. They are given in Table VI. and fig. 4.

In testing the quality of an optical instrument, say an object-glass, two different methods may be used. The first is to test all the errors (aberrations) separately; the second to test the instrument in its whole, for example, by the determination of its resolving power. The first method as relating to the linear aberrations or to the aberrations of wave-front does not give a definite answer to the general performance of the instrument, as we do not know till now the accurate (numeric) relations between the aberrations taken as a whole and the resolving power of the instrument.

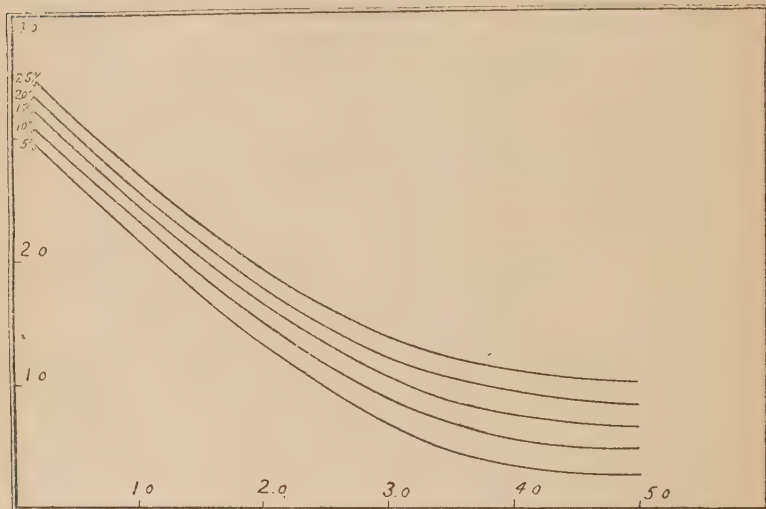
TABLE VI.

	D = 0.2.	0.4.	0.6.	0.8.	1.0.	1.2.	1.4.	1.6.	1.8.	2.0.	2.2.	2.4.	2.6.	2.8.	3.0.
5 per cent. ...	2.95	2.75	2.56	2.36	2.18	2.00	1.84	1.67	1.51	1.34	1.20	1.06	.92	.79	.68
10 "	3.07	2.89	2.70	2.52	2.34	2.18	1.99	1.82	1.67	1.52	1.37	1.24	1.11	1.00	.90
15 "	3.21	3.03	2.82	2.65	2.47	2.30	2.13	1.97	1.82	1.68	1.57	1.40	1.28	1.17	1.06
20 "	3.33	3.14	2.95	2.77	2.59	2.42	2.25	2.11	1.94	1.80	1.68	1.54	1.42	1.32	1.22
25 "	3.46	3.25	3.08	2.90	2.72	2.56	2.38	2.24	2.08	1.96	1.83	1.70	1.58	1.48	1.39

	D = 3.2.	3.4.	3.6.	3.8.	4.0.	4.2.	4.4.	4.6.	4.8.	5.0.	6.0.	7.0.	8.0.	9.0.	10.0.
5 per cent.59	.50	.44	.38	.34	.32	.30	.29	.28	.27	.26	.24	.22	.21	.21
10 "	.80	.72	.67	.60	.56	.53	.51	.49	.48	.47	.44	.42	.40	.39	.39
15 "	.97	.90	.85	.80	.76	.73	.70	.68	.67	.65	.62	.60	.58	.58	.58
20 "	1.15	1.08	1.02	.98	.94	.91	.89	.86	.84	.83	.80	.79	.77	.76	.76
25 "	1.32	1.25	1.19	1.15	1.13	1.10	1.06	1.04	1.02	1.02	.99	.97	.96	.95	.95

The second method may give a general characteristic of the instrument, but only in the case of our knowledge of theoretical value of resolving power of, an *ideal object-glass*, as by the discrepancies (found experimentally) from this theoretical value it is possible to judge about the quality of the instrument. The theoretical values of resolving power of an *object-glass* based on the distribution of

Fig. 4.



illumination in a diffraction image for several luminous objects (two points, two planes, two lines) were investigated for this purpose. But, as said before, so as to state the *practical limit of resolution of an ideal object-glass* for such objects, it is necessary to know the contrast sensibility of the eye for the distribution of illumination like that in a diffraction image of two luminous objects and for a given brightness of them.

The author is much indebted to Mr. G. Wahrlich for execution of the drawings and for assistance in computations.

CVIII. *The Complexity of the Elements.*—Part I. *Elements of Odd Atomic Number* *. By A. S. RUSSELL, M.A., D.Sc., Student and Tutor of Christ Church, Oxford †.

§ 1. *Introduction.*

IN a previous paper ‡ the writer has shown that there is evidence for the possible existence of four disintegration series, the members of which have atomic weights given respectively by $4n+3$, $4n+2$, $4n+1$, and $4n$, where n is an integer. In that paper reasons are given for supposing that the first of these is the actinium series, the second the uranium series, the third a hypothetical series the end products of which may be bismuth ($a=209$) and thallium ($a=205$), and the fourth the thorium series. The properties of a group of radioactive isotopes were compared with those of a group of isotopes belonging to an inactive element and points of similarity and of difference discussed. The purpose of this paper is to elaborate this comparison in order to see what light the known properties of isotopes of the radio-elements may throw on those of inactive elements. It has been found possible, by boldly applying to the inactive elements what is known to exist for, or is hinted at by the radio-elements, to predict with some degree of confidence the mass-numbers of both even and odd elements over a wide, although not a complete range of atomic numbers. In this work great use has been made of the accepted atomic weights of the elements but, as far as possible, I have tried to develop the results independently of Aston's published experimental values of mass-numbers so that these latter may be made use of as a check on the predictions. In this paper elements of odd atomic number will be chiefly considered.

§ 2. *Radioactive and common Isotopes.*

The disintegration series as derived in my earlier paper are set out in fig. 1 to show the relation of atomic weights to atomic number and to illustrate the points to be developed.

It may be seen from the figure that if bodies which emit α -particles only are considered (end-products must count as

* A preliminary account of this work appeared in 'Nature,' vol. cxii. p. 588 and p. 619 (1923).

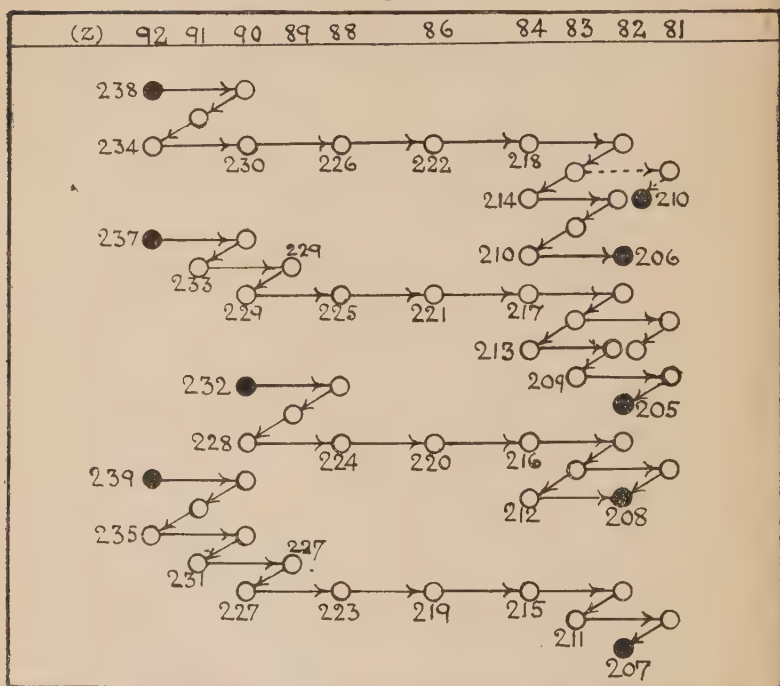
† Communicated by the Author.

‡ A. S. Russell, Phil. Mag. xvi. p. 642 (1923).

α -rayers, but not bodies which branch principally in the β -direction):—

- (a) The maximum difference in mass-number of any group of isotopes is 8. This holds for the inactive elements investigated by the positive-ray method, the lowest element to show this maximum difference being selenium ($z=34$).

Fig. 1.



- (b) Elements of odd atomic number may have two odd mass-numbers only and these differ by two units. This has been pointed out by Aston* as holding for all elements of odd atomic number so far investigated by him.
- (c) Elements of even atomic number may have both odd and even mass-numbers and, if elements 84 and 92 be excepted, the number of the former is limited to two and their difference is two units. There are no definite exceptions to this among Aston's results. The most complex elements so far investigated by him, numbers 80, 54, 50, and 36, have each two odd isotopes differing by two units of mass-number.

* F. W. Aston, *Phil. Mag.* xxxix. p. 611; xl. p. 628; xlii. p. 140; xlii. p. 436.

The reason for excluding β -rayers in making these comparisons is that these bodies have relatively quick periods compared with their α -ray isotopes, so that if a radio-element was regarded as a common element containing isotopes the β -ray isotopes would be present in negligibly small proportion. This exclusion, however, does not allow of the existence of isobares amongst radioactive isotopes and, in consequence, makes the radio-elements appear less complex than the inactive elements, for these possess, as Aston has shown, isobares. This reasonable objection is removed with respect to even mass-numbers and, at the same time, the comparisons above remain as before, if the second β -rayer of a change involving the expulsion of two successive β -particles alone be excluded. The second β -rayer has invariably a smaller period than the first, the ratio of the periods of the two varying from 3.2×10^{-5} to 0.73.

It may be noted also that in the "repeated" change, that of radium A ($a=218$) to radium Ω' ($a=206$), the periods of each body in the series from radium D ($a=210$) to radium Ω' is much greater than that of its isotope between radium B ($a=214$) to radium D. This is the only certain change of this type in radioactivity, but if it were typical then the element, $z=91$, $a=231$, would have a greater period than $z=91$, $a=233$, and $z=83$, $a=209$, than $z=83$, $a=211$. This is the merest hint that the lighter of the two isotopes of an odd element is the more prevalent in Nature. On examining the values of the atomic weights of the inactive elements and such results as Aston has already obtained experimentally, it is found that this rule appears to hold without exception for the isotopes of inactive elements of odd atomic number. There are indeed elements where the proportion of heavier to lighter isotope is nearly unity but none in which unity is exceeded.

There are two more relations hinted at rather than clearly expressed by the numbers in fig. 1. The first is that the difference $\frac{a_1 + a_2}{2} - 2z$, where a_1 and a_2 are the mass-numbers of an odd element of atomic number z , applies to one element only; the second is that the difference $a - 2z$, where a is the odd mass-number of an even element of atomic number z , applies to one element only. Neither of these holds for the radio-elements as arranged in fig. 1, as Tables I. and II. show. This may be because these relations are false ones; or it may be that not all the elements postulated in fig. 1 actually exist. Nevertheless, the relations will be made use of in what follows as empirical ones, as relations

likely to provide a minimum amount of information concerning the mass-numbers of inactive elements of odd atomic number. Should this information be consistent with experimental results these relations may be applied strictly to the radio-elements.

TABLE I.

$\frac{a_1+a_2}{2} - 2z.$	$a_1.$	$a_2.$	$z.$
52	235	237	92
50	231	233	91
48	227	229	90
48	223	225	88
48	219	221	86
46	213	215	84
44	209	211	83
42	205	207	82

TABLE II.

$a - 2z.$	$a.$	$z.$
53	237	92
51	235	92
49	229	90
49	225	88
49	221	86
49	217	84
47	227	90
47	223	88
47	219	86
47	215	84
45	213	84
43	207	82
41	205	82

There is one respect at least in which the isotopes postulated in fig. 1 differ from those experimentally discovered by Aston. In all Aston's results* the odd isotopes of even elements are never the lowest in mass-number; according to fig. 1, however, odd isotopes are the lowest in three cases out of six. It would indeed appear that by analogy with Aston's results the hypothetical series is the actinium series and *vice versa*. It will be shown in a later paper, partly from experimental work and partly by the arguments of this paper, that this need not be so and that the contradiction between the two sets of results is more apparent than real.

There is thus a *prima facie* case for considering that radioactive isotopes are not markedly different from those of the inactive elements.

* F. W. Aston, Phil. Mag. xlv. p. 944 (1923).

It is clear from fig. 1 that a succession of α -particles and the succession $\alpha, \beta, \beta, \alpha$ are the characteristic radioactive changes of the two even series, and a succession of α -particles and probably the succession $\alpha, \beta, \alpha, \beta$ of the odd series. If it were imagined that these series continued below the so-called end-products, it is simplest to assume that the former set of changes are characteristic of elements of even atomic number and the latter of elements of odd atomic number. This, however, is not enough. The two odd series in fig. 1 do not pass from elements 89 to 83 through the odd elements 87 and 85, but, like the even series, through 88, 86, and 84. But if these odd series are assumed to continue through inactive elements both or at least one must be imagined as passing through odd elements only, since the even series pass through the even elements, and there are as many odd elements in Nature as even. In other words, if the isotopes of the inactive elements are being or have been produced by the processes of radioactivity it might be that the odd, radioactive series are not wholly characteristic because of the non-existence (supposedly due to instability) of elements 93, 87, and 85.

The main assumption above does not necessarily involve the production of the inactive elements by disintegration, but merely states that the isotopes of inactive elements, however they have been produced in the past, are stable or unstable dispositions according as they would be presumed stable or unstable on radioactive evidence. This is reasonable and not new*. If it leads to a satisfactory agreement between experimental and predicted values of the mass-numbers of the inactive elements, it may reveal a law which may throw light on the stability and composition of the nucleus and the relation of the radio-elements to the inactive elements.

§ 3. Possible Mass-Numbers of Odd Elements.

The odd elements will first be considered. Bismuth ($z=83, a=209$) is taken as datum and the assumption made that odd elements of the form $z=4n+3$ have been genetically connected like the members of a radioactive series, α - and β -like particles only having been expelled. The difference in a corresponding to 4 in z may thus be 8, 12, or

* W. Kossel, *Phys. Zeit.* xx. p. 265 (1919). W. D. Harkins, *Phys. Review*, xv. p. 73 (1920); *J. Amer. Chem. Soc.* xlii. p. 1956 (1920); *Phil. Mag.* xlii. p. 305 (1921). S. Meyer, *Wien. Ber.* cxxiv. p. 249 (1915). A. van den Broek, *Phys. Zeit.* xvii. p. 260 (1916); xxi. p. 337 (1920).

16 only. The atomic weight of the elements is used as a guide in deciding which of these differences of a is the most probable. The same is done for elements of the form $z=4n+1$, thallium ($z=81$, $a=203$) being taken as datum. For simplicity I make the limiting assumption that the differences in a are the same for corresponding members of the two series, *i. e.* that both series have disintegrated similarly.

The results of this are tabulated in Table III.

TABLE III.

Range $z=85$ to $z=19$.

z .	a .	δa .	a .	z .
83	(209)	12	215	85
79	197	12	(203)	81
75	185	8	191	77
71	177	12	183	73
67	165	12	171	69
63	153	12	159	65
59	141	8	147	61
55	133	12	139	57
51	121	12	127	53
47	109	12	115	49
43	97	8	103	45
39	89	8	95	41
35	81	12	87	37
31	69	12	75	33
27	57	8	63	29
23	49	8	55	25
19	41		47	21

I next assume that each odd element may have like the radioactive elements (§ 2 (b)) two isotopes whose mass-numbers differ by two units. In some cases the second mass-number may be one of two alternatives; in others there is no alternative. (For values of $z=4n+3$ there are alternatives only when the difference between the values of a for $z=4n+5$ and $z=4n+1$ in Table III. is 12.) These possibilities are included in Table IV.

TABLE IV.

Possible mass-numbers of elements of atomic number
 $4n+3$ over the range of z 83 to 19.

z .	First isotope.	Possible second isotope.
83	209	207, 211
79	197	195, 199
75	185	187
71	177	175, 179
67	165	163, 167
63	153	151, 155
59	141	143
55	133	131, 135
51	121	119, 123
47	109	107, 111
43	97	99
39	89	91
35	81	79, 83
31	69	67, 71
27	57	59
23	49	51
19	41	39, 43

The same assumption is made in compiling Table V. Here, for values of $z=4n+1$, there are alternatives for the second isotope only when the difference between the values of a for $z=4n+3$ and $z=4n-1$ in Table III. is 12.

TABLE V.

Possible mass-numbers of elements of atomic number
 $4n+1$ over the range of z 81 to 21.

z .	First isotope.	Possible second isotope.
81	203	205, 201
77	191	193, 189
73	183	181
69	171	173, 169
65	159	161, 157
61	147	149, 145
57	139	137
53	127	129, 125
49	115	117, 113
45	103	105, 101
41	95	93
37	87	85
33	75	77, 73
29	63	65, 61
25	55	53
21	47	45

It is now possible to decide which of the odd elements are simple and which double and to find out the mass-numbers of their isotopes in the following way:—

- (1) Certain odd mass-numbers not likely to belong to odd elements, *i. e.* not in the second columns of Tables IV. and V., are assigned to certain even elements according to a method described in § 4.
- (2) These and other odd mass-numbers are assigned to certain even and odd elements, which are believed to have each *two* odd isotopes according to a different but analogous plan described in § 5. It is then found that certain elements should be simple and they are assigned the mass-numbers nearest to the accepted values of their atomic weights.
- (3) A principle, namely that the complexity of the elements over a certain range is a periodic function of the atomic number, is then revealed, and this helps to coordinate and check results. This is described in § 6.

§ 4. *Elements of Even Atomic Number containing an Isotope of Odd Mass-Number.*

These elements can be picked out from the others by making the following assumptions:—

- (1) that no odd isobares exist (see § 2);
- (2) that the mass-numbers in the second column of both Tables IV. and V. belong to the elements assigned to them there;
- (3) that the difference $a - 2z$ where a is the odd isotope of an even element z applies to one element only. This difference is the so-called "excess weight" of an element, and is called by Harkins the "Isotopic Number." This assumption was suggested by radioactive evidence and has been discussed in § 2 above.

The values of z and a corresponding to values of $a - 2z$ from 3 to 43, deduced from these assumptions, are included in Table VI.

It is seen that there are alternative values of z and a for values of $a - 2z$ of 35, 23, and 11 only. It is decided in the next section that the lowest values of a and z are the only possible alternatives for each of these differences, and this is indicated already in two of the three cases by Aston's

experimental values. Values of $a - 2z$ less than 9 appear to have no elements to which they can be allotted. This is probably due to the fact that none of the elements in this region of atomic number is sufficiently complex to have an odd isotope. As none of the four values of a , inconsistent with Aston's experimental work, apparently belongs to any element at all, it is possible that if these mass-numbers existed they would belong to the element to which they are assigned in the table.

TABLE VI.

Even elements with odd isotopes.

$a - 2z$.	z .	a .	Experimental evidence in confirmation.
43	82	207	Radioactivity ($\text{Ac } \Omega$).
41	82	205	—
39	80	199	Aston *.
37	76	189	—
35	76	187	—
	72	179	—
	70	175	—
33	70	173	—
31	66	163	—
29	66	161	—
27	64	155	—
25	60	145	—
23	60	143	—
	56	135	—
	54	131	Aston.
21	54	129	Aston.
19	50	119	Aston.
17	50	117	Aston.
15	48	111	—
13	44	101	—
11	44	99	—
	40	91	—
	36	83	Aston.
9	34	77	—
7	30	67	Aston's results are against all these possibilities.
5	28	61	
	24	53	
3	20	43	

* F. W. Aston, *Phil. Mag.* xlv. p. 944 (1923).

§ 5. *Elements containing Two Isotopes of Odd Mass-Number.*

These elements are distinguished by making use of the assumption mentioned in § 2 that if z be the atomic weight of any element having two odd mass-numbers a_1 and a_2 then the difference $\frac{a_1 + a_2}{2} - 2z$ is a number which applies to one element only.

On this assumption and making use of previous tables, Table VII. is compiled.

TABLE VII.

Elements having two isotopes of odd mass-number.

$\frac{a_1 + a_2}{2} - 2z.$	$a_1.$	$a_2.$	$z.$
42	205	207	82
40	—	—	—
38	195	197	79
36	181	183	73
34	173	175	70
32	165	167	67
30	161	163	66
28	—	—	—
26	151	153	63
24	137	139	57
22	129	131	54
20	121	123	51
18	117	119	50
16	—	—	—
14	107	109	47
12	93	95	41
12	85	87	37
10	79	81	35
8	69	71	31
6	63	65	29
4	45?	47?	21?
2	{ 39	41	19
	{ 35	37	17

Below atomic number 37 a certain regularity which exists above it disappears. This regularity is that for a difference of $z=16$ there is a difference of $a=44$. Below this atomic number also the rule is sometimes contradicted by the experimental evidence.

It is seen that values of $\frac{a_1 + a_2}{2} - 2z$ equal to 40, 28, 16

(and possibly also 4) appear to have no corresponding value of z , for the only possibilities in these cases are :

$\frac{a_1+a_2}{2} - 2z.$	$a_1.$	$a_2.$	$z.$
40	197	199	79
28	157	159	65
16	113	115	49
4	45	47	21

It is decided later (§ 8) that mass-numbers 113 and 157 are unstable ; 199 has already been assigned to element 80, to which it has been proved by experiment to belong and so cannot belong also to element 79. Nothing can be said at this stage for the difference 4, since it may belong to an element 21 in the irregular part of Table VII. It is concluded, therefore, that the differences 40, 28, and 16 cannot be assigned to any element.

It is of course possible that the assumption that the differences in the first column of Table VII. apply each to one element only is unnecessarily simple. It might be that certain differences apply to two or more elements according to some regular principle, but this does not appear likely unless odd isobares exist, and will not be considered here as the problem to be attacked is sufficiently complex when stated in its simplest terms.

There are two consequences of Table VII. The first is that all elements in the range $z=82$ to $z=37$ not included in the last column are simple, if odd, and do not possess more than one odd isotope, if even. Simple odd elements are therefore atomic numbers 39, 43, 45, 49, 53, 55, 59, 61, 65, 69, 71, 75, 77, and 81. These are tabulated with their mass-numbers below. The second is that two of the three differences in the first column of Table VI., which there may be assigned to one of three elements, are now seen to be assignable only to the element of lowest atomic number in each case. By analogy the third difference in that table is also assigned to the element of lowest atomic number. It is found later (§ 8) that the rejected mass-numbers cannot be assigned to any element. Table VI. should in consequence be amended to read as follows :—

$a-2z.$	$z.$	$a.$	Experimental evidence in confirmation.
35	70	175	—
23	54	131	Aston.
11	36	83	Aston.

In Table VIII. is given for reference the mass-numbers of the simple odd elements. They are the "first isotopes" of the tables referred to, except in one case where the second isotope is in agreement with the evidence of atomic weight.

TABLE VIII.

Simple elements of odd atomic number.

<i>z.</i>	<i>a.</i>	Reference to Table.	Experimental evidence in confirmation.
81	203	V.	At.-wt. is against*.
77	193	V.	—
75	185	IV.	—
71	177	IV.	—
69	171	V.	—
65	159	V.	—
61	147 or 149	V.	—
59	141	IV.	—
55	133	IV.	Aston; At.-wt.
53	127	V.	Aston; At.-wt.
49	115	V.	Aston.
45	103	V.	—
43	97 or 99	IV.	—
39	89	IV.	Aston.

* A recent determination by Hönigschmid and Kothe gives the value 204.39 (K. Fajans, 'Radioactivity,' p. 129, 1923).

§ 6. Complexity a Periodic Function of the Atomic Number.

In Table IX. the atomic numbers of even elements are tabulated according to their arrangement in Tables VI. (as amended) and VII.

TABLE IX.

Atomic numbers of elements with and without isotopes of odd mass-number.

No odd isotopes.				One odd isotope.			Two odd isotopes.	
78	74	72	68	76	80		82	
62	58	56	52	60	64		70	66
46	42	40	38	44	48	36	54	50

Elements 36 and 38 alone do not conform to the grouping by difference of 16. But over the range of $z=40$ to $z=82$

inclusive the complexity of even elements with respect to odd isotopes is a periodic function of the atomic number 16.

The simple odd elements included in Table VIII. may also be grouped in a similar way, thus :

81	77	75	71	69
65	61	59	55	53
49	45	43	39	

The odd elements having two isotopes included in Table VII. may also be similarly grouped :

79	73	67
63	57	51
47	41	

Thus the complexity of odd elements over the range of $z=41$ to $z=81$ appears also to be a periodic function of the atomic number 16. It is found by inspection that the difference in a corresponding to that in z of 16 is usually 44. For heavy elements it may be 48; for light ones 40 and even occasionally 36. These differences are all of the form $4n$, and their rôle is readily understandable from work on radioactivity.

To illustrate the periodicity of the complexity of the odd elements the mass-numbers are arranged in 8 groups of atomic numbers having the forms $16n+1$, $16n+3$, $16n+5$, $16n+7$, $16n+9$, $16n+11$, $16n+13$, $16n+15$. These are set forth in Table X. The mass-numbers of the elements in the range which is chiefly being considered ($z=39$ to $z=83$) are derived from the tables to which references are made. The investigation is, however, pushed into the region both of higher and lower elements by making use in the former case of the supposed mass-numbers of the stable radio-elements, and, in the latter, by inserting the known mass-numbers of the lowest element of each series, and linking them with the mass-numbers given in the tables of this paper through the mass-numbers of elements that have not been considered here. In this manner new information is obtained, and in Table X. such information is indicated by absence of reference in the column for references. This will be considered in Table XI. below.

In each column of mass-numbers in the table the differences are of the form $4n$ and are, as has been said above, generally 44 or 40.

TABLE X.

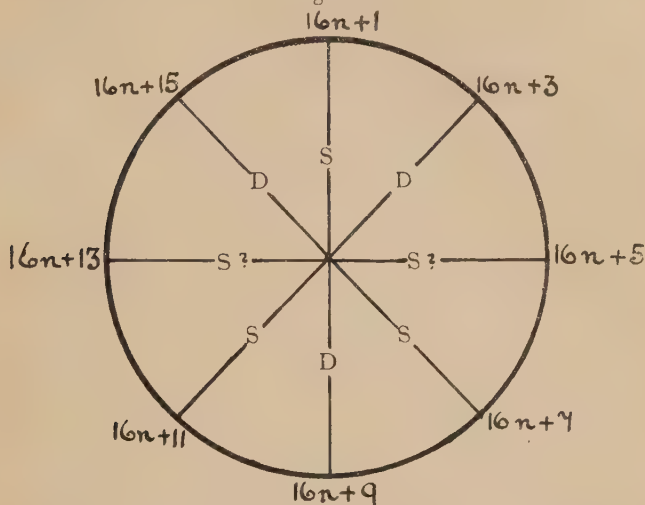
Atomic Number. <i>z.</i>	Mass- Number. <i>a</i> ₁ <i>a</i> ₂ .		Reference to Tables.	Atomic Number. <i>z.</i>	Mass- Number. <i>a</i> ₁ <i>a</i> ₂ .		Reference to Tables.
(16 <i>n</i> +1)				(16 <i>n</i> +9)			
81	203	—	VIII.	89	229	227	
65	159	—	VIII.	73	181	183	VII.
49	115	—	VIII.	57	137	139	VII.
33	75	—		41	93	95	VII.
17	35 or 37			25	—	55	
1	—	1	At.-wt.	9	—	19	At.-wt.
(16 <i>n</i> +3)				(16 <i>n</i> +11)			
83	211	209		91	233	—	
67	167	165	VII.	75	185	—	VIII.
51	123	121	VII.	59	141	—	VIII.
35	79	81		43	97 or 99		VIII.
19	39	41		27	—	59	
3	7	5(? 6)	At.-wt.	11	—	23	At.-wt.
(16 <i>n</i> +5)				(16 <i>n</i> +13)			
85	217?	215?		77	—	193	VIII.
69	—	171	VIII.	61	147 or 149		VIII.
53	—	127	VIII.	45	103	—	VIII.
37	85 or 87			29	63	65	
21	45 or 47			13	27	—	At.-wt.
5	9(10)	11	At.-wt.	—	—	—	
(16 <i>n</i> +7)				(16 <i>n</i> +15)			
87	221	—		79	195	197	VII.
71	177	—	VIII.	63	151	153	VII.
55	133	—	VIII.	47	107	109	VII.
39	89	—	VIII.	31	71	69	
23	—	51		15	31	—	At.-wt.
7	—	15(14)	At.-wt.	—	—	—	

In the above table the three mass-numbers in brackets are the experimental values; those beside them are those to be expected from the sequence of the mass-numbers of the table to which they belong. It is obvious, however, that elements 3 and 5 cannot have mass-numbers less than 6 and 10 respectively. The discrepancies arise probably because the considerations which led to the prediction of these mass-numbers do not apply to the lightest elements. It is seen that series $16n+1$, $16n+7$, and $16n+11$ are simple. Series $16n+3$ and $16n+15$ are double and $16n+9$ is chiefly double. Series $16n+5$ and $16n+13$ are more irregular than the others but are more simple than double, especially $16n+13$.

A relation between these series is shown in fig. 2. S represents "simple" or "chiefly simple," D "double or "chiefly double." The two irregular series are characterized as S?. The line joining $16n+13$ and $16n+5$ divides the figures into opposing halves. About the line joining $16n+1$ and $16n+9$ the figure is symmetrical. A simple relation of this kind

is to be expected if the complexity of the elements be a periodic function of the atomic number.

Fig. 2.



The new information deducible from the arrangement of Table X. and from fig. 2 is set forth with notes in Table XI.

TABLE XI.

z .	a_1 .	a_2 .	Notes on the Mass-Numbers.	Authority confirming a_1 and a_2 as set out.
83	211	209	$\left\{ \begin{array}{l} a_1 \text{ is probably radioactive, so} \\ \text{that the mass-number of} \\ \text{the element is that of } a_2. \end{array} \right\}$	Radioactivity.
69	171	---	This value appears more probable than 169.	
37	85	87	$\left\{ \begin{array}{l} \text{Nothing can be said as series} \\ 16n+5 \text{ is the most irregu-} \\ \text{lar of all.} \end{array} \right\}$	Aston.
35	79	81	Both are certain.	Aston.
33	75	—	a_2 could only be 77 which belongs to $z=34$.	Aston.
31	69	71	Both are certain.	Aston.
29	63	65	The lower is alone certain.	Aston.
27	59	—	Simple element.	Aston.
23	51	—	Simple element.	Aston.
21	45	—	$\left\{ \begin{array}{l} \text{Nothing can be said as series} \\ 16n+5 \text{ is the most irregu-} \\ \text{lar of all.} \end{array} \right\}$	Aston.
19	39	41	Both are certain.	Aston.
17	35	37	$\left\{ \begin{array}{l} \text{Only one is certain, and it} \\ \text{cannot be said which is} \\ \text{the more probable.} \end{array} \right\}$	Aston.

§ 7. *Discussion of Results.*

When the mass-numbers which have been calculated by the methods described in this paper and tabulated in the various tables are compared with those actually determined by the positive-ray method or those to be anticipated from reliable determinations of atomic weights, it is seen that discrepancies are most apparent when elements of low atomic number are considered. These are seen in Table II. for the mass-numbers of elements 27 and 23 ; in Table VI. for those of elements 30, 28, 24, and 20 ; in Table VII. for those of elements 37, 29, 21, and 17 ; in Table IX. for elements 38 and 36.

It is clear, then, that if the principle, that the complexity of the elements is a periodic function of atomic number, holds, it holds over a range that does not include the atomic numbers above, *i. e.* it holds over the range of atomic numbers 39 to 83. It will be shown in a subsequent paper that experimental evidence supports, with qualification, the extension of this range to atomic number 92. The obvious cause of failure of the principle to embrace the lower atomic numbers is the relative simplicity of these elements. This is partly borne out by Aston's results. He finds that the largest possible difference in the mass-numbers of isotopes rises from 1 first shown by element 3, to 8 first shown by element 34, being 2 first for element 10, 4 first for element 18, and 6 first for element 32. Since the difference of 8 apparently is not exceeded even by the radioactive elements, it may be taken provisionally as a measure of the possible complexity of an element. There is thus a limit to possible complexity which cannot be attained by elements below number 34. From 34 to 92 there is, in consequence, a chance that some connecting principle may hold.

There are some results in Table X. which appear to be wrong.

(1) Thallium $z=81$ appears to have $a=203$ only. The atomic weight of thallium, which is 204.0 in the International Table, and also fig. 1 suggest very strongly, however, a second isotope at 205. But if this be so, an exception arises both to the simplicity of elements of the series $16n+1$ and also to the existence of isobares of odd mass-number, for there is a probability that element 82 (lead) has an isotope of this mass-number (Table VII.).

(2) It is not likely that gold ($z=79$) has a second isotope at 195, as the table suggests, in any detectable concentration, as the accepted atomic weight is 197.2. It is also exceptional

for the higher of two mass-numbers to be the more abundant. The mass-number 199, on the other hand, is assigned to mercury ($z=80$) (Table VI.) which has received experimental confirmation by Aston, and is consequently ruled out unless odd isobares exist. The difficulty appears best resolved by regarding gold as 197 with a small, probably, negligible amount of 195, *i. e.* practically a simple element.

(3) Lanthanum ($z=57$) if double is also more abundant in its higher isotope 139 than in its lower 137. The atomic weight in the International Table is 139.0 although several experimental determinations fall below this value. The atomic weight evidence against the simplicity of this element is consequently conflicting.

It is a possible inference from (2) and (3) that some elements which appear to be double in form may really be single. Such have atomic weights which approximate to the higher of their possible mass-numbers. In all such cases the lower cannot apparently be assigned to another element and is, in consequence, a missing mass-number.

It will be shown in a later paper which deals with the mass-numbers of even elements that periodicity is shown there more strikingly than with odd elements. Periodicity is, however, least in agreement with the apparent facts in the series of elements whose mass-numbers are of the form $16n+15$ and $16n+13$. These series correspond, by analogy, with the groups of the Periodic System (such as groups VI. A and I. B) in which the chemical resemblances of members are by no means marked. It is impossible to predict with any confidence the complexity of elements of this series, and the uncertainty is likelier to obtain for the lighter than for the heavier members.

It is noticeable that several elements of the double series $16n+3$ and $16n+15$ have isotopes in nearly equal proportion as judged from evidence of atomic weight: elements 35 and 51 in the former series, and 31, 47, 63 (not 79, however, but perhaps 81) in the latter. In the third double series $16n+9$, however, the mean atomic mass of the elements approximates to that of one of its isotopes.

§ 8. Missing Odd Mass-Numbers.

There appears to be three definite groups of missing odd mass-numbers, and these may be derived from the numbers given in Table X. Each of the series $16n+1$, $16n+7$, and $16n+11$ is simple. The inclusion of certain mass-numbers

in each would make them double. As none of these mass-numbers are assignable to elements of even number (Table VI.) they are presumably missing. The numbers so derived are included in Table XII. and fall into three complete groups. The numbers derived from the incomplete, simple series $16n+5$ and $16n+13$ are also given.

TABLE XII.

Groups of missing odd mass-numbers.

Series from which numbers are derived (Table X.).	Derived missing Mass-Numbers.
$16n+1$	201, 157, 113, 73, 33
$16n+7$	223, 179, 135, 91, 49, 13 53, 17
$16n+11$	231, 187, 149, 99, 57, 21 61, 25
$16n+5$	169, 125
$16n+13$	191, 149 or 147, 105, 29

Of these mass-numbers two only, 25 and 29, have been proved to exist. They fall, however, outside the range that is chiefly being considered here, and, moreover, they do not belong to odd elements but to even ones. The following is the list of missing odd mass-numbers compiled from Aston's work : 201, 73, 67, 61, 57, 53, 49, 47, 43, 33, 21, 17, 15, 13, 5, and 3. All of these except 67, 47, 43, 15, 5, and 3 are included in the table above. 67 and 43 are not derived from, but accounted for, by Table VI. These would be the isotopes of elements 30 and 20 respectively if the latter were sufficiently complex to include odd isotopes. These, unlike the others, are not members of a series of missing mass-numbers. Some explanation involving the simplicity of the very lightest elements might account for the non-existence of 3, 5, and 15. (Analogies would make 15 not 14 the single mass-number of nitrogen.) There remains of Aston's list mass-number 47. Table VII. leaves the question of its existence open ; for, by analogy with the mass-numbers of higher elements, scandium ($z=21$) should be single, but this element occurs at a part of the table which does not conform to the simplicity of the greater part. Table X. also is indecisive although the balance of probability is in favour of the existence of 47. There are, however, three further

considerations. The first is that the occurrence of two consecutive, missing, odd, mass-numbers is rare; the second that the existence of a mass-number, of the form $4n + 3$, one unit less than that of a simple element seems to be the rule for all mass-numbers above 47. On the other hand, if scandium be double it is one of three consecutive double odd elements, and this appears to be exceptional. On the whole the evidence from analogy is in favour of the existence of 47. Aston's experimental work is not decisively against this view.

A second paper will deal with the isotopes of elements of even atomic number.

§ 9. Summary.

(1) By analogies with and deductions from the properties of the radioactive elements, it is concluded that over a range of atomic number 39 to 83 the complexity of elements appears to be a periodic function of the atomic number 16. The corresponding mass-number is usually 44 or 40.

(2) Elements of odd atomic number are principally considered. It is deduced that the following should be simple; atomic numbers 1, 7, 11, 17, 23, 27, 33, 39, 43, 45, 49, 53, 55, 59, 61, 65, 69, 71, 75, 77, 81, 87, 91. Probably also 57, 79, and 83.

(3) The following should consist of two isotopes: atomic numbers 3, 19, 31, 35, 41, 47, 51, 63, 67, 73, and 89. Numbers 57, 79, and 83 are double in form.

(4) It is not possible to pronounce definitely on the complexity of elements 5, 9, 13, 15, 21, 25, 29, 37, and 85 from these considerations alone.

(5) The elements mentioned in (2) and (3) with their atomic weights, mass-numbers, etc., are given for reference in Table XIII. A blank space denotes there is no information available, a dash that there is no mass-number. The reason for the brackets around a few of the numbers has been explained in § 7.

(6) The following mass-numbers belong to even elements: 77 to 34, 83 to 36, 101 to 44, 111 to 48, 117 and 119 to 50, 129 and 131 to 54, 145 to 60, 155 to 64, 161 and 163 to 66, 173 and 175 to 70, 189 to 76, 199 to 80, 205 and 207 to 82.

TABLE XIII.

Element.	<i>z.</i>	Atomic Weight.	Expected Mass-Numbers.		Experimental Mass-Numbers.	
Pa	91		233	—		
Ac	89		227	229		
	87		221	—		
Bi	83	209·0	209	(211)		
Tl	81	204·0	203	—	203 ?	205 ?
Au	79	197·2	197	(195)		
Ir	77	193·0	193	—		
	75		185	—		
Ta	73	181·5	181	183		
Lu	71	175	177	—		
Tm	69	168·5	171	—		
Ho	67	163·5	165	167		
Tb	65	159·2	159	—		
Eu	63	152·0	151	153		
	61		147	or 149		
Pr	59	140·9	141	—		
La	57	139·0	139	(137)		
Cs	55	132·81	133	—	133	—
I	53	126·92	127	—	127	—
Sb	51	121·8	121	123	121	123
In	49	114·8	115	—	115	—
Ag	47	107·88	107	109	107	109
Rh	45	102·9	103	—		
	43		97	—		
Nb	41	93·1	93	95		
Y	39	89·33	89	—	89	—
Br	35	79·92	79	81	79	81
As	33	74·96	75	—	75	—
Ga	31	70·1	69	71	69	71
Co	27	58·97	59	—	59	—
V	23	51	51	—	51	—
K	19	39·1	39	41	39	41
Cl	17	35·45	35	—	35	37
Na	11	23·00	23	—	23	—
N	7	14·01	15	—	14	—
Li	3	6·94	5	7	6	7
H	1	1·008	1	—	1	—

(7) The following mass-numbers appear to be missing on analysis: 13, 17, 21, 25, 29, 33, 49, 53, 57, 61, 73, 91, 99, 105, 113, 125, 135, 143, 149 or 147, 157, 169, 179, 187, 191, 201, 223, 231. Probably also 137, 195 and 211.

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February 1924.

CIX. *Elastic Impact of Pianoforte Hammer.*
 By S. BHARGAVA, M.Sc., and R. N. GHOSH, M.Sc.*

[Plate VIII.]

I.

HELMHOLTZ built up his classical theory of the vibration of pianoforte strings on the assumption that the law of pressure was given by $F \sin pt$ †. He regarded the hammer as elastic, and the duration of impact as small compared with the period of vibration of the string. Later on, Kaufmann ‡ showed that the duration of impact was comparable to the period of vibration of the string, and he based his theory on the assumption that the hammer was hard. Professor Raman and Mr. Banerji § conjointly and Mr. Das || separately have recently extended the theory of the hard hammer, and they have given a method for calculating the duration of impact in the general case when the striking-point lies anywhere in the string. The present paper attempts to develop the theory of impact of the elastic hammer when the duration of impact is comparable to the period of vibration of the string. The theory has been put to test by experiments with hammers and strings actually used in the piano.

II.

Ignoring for the present the shape of the hammer, and its effect on the duration of impact in accordance with the theory of Hertz ¶, we simplify the problem by considering that the hammer is made up of a hard mass backed by a weightless spring, which together impinge upon the string. The shape of the hammer actually used has been chosen in order to increase the effective area of contact, which has the effect of increasing the convergency of partials. With these assumptions we shall now proceed to develop the

* Communicated by the Authors. Read before the Indian Science Congress, January 1923.

† 'Sensation of Tone' (Ellis translation), p. 380.

‡ *Ann. der Physik*, liv. p. 675 (1895).

§ *Proc. Roy. Soc.* xcvii. p. 99.

|| *Proc. Ind. Assoc.* vii. p. 13.

¶ Love's 'Theory of Elasticity,' Art. 139 (1906 ed.).

1142 Messrs. S. Bhargava and R. N. Ghosh on
theory in a few cases. The symbols used in the paper
are :

- M = Mass of hammer.
 ρ = Mass of string per cm.
 y = Displacement of the string.
 y_0 = Displacement of the centre of mass of the
 hammer.
 ξ = Compression of the spring.
 μ = Strength of the spring.
 l = Length of the string.
 T = Tension of the string.
 C = Velocity of transverse wave in the string.
 α = Distance of the striking-point from the nearer
 extremity.
 t = The time.

Case I.

Length of the string finite.

Let us consider a theoretical case in which the mass
of the whole string is supposed to be collected at the
point of impact which divides the string into two portions,
 α and $l-\alpha$. The equation of motion of the hammer is
given by

$$\frac{M d^2 y_0}{dt^2} = -p, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where p represents the pressure between the hammer and
the mass ρl .

$$p = \mu \xi, \quad y_0 = y + \xi. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The equation of motion of the mass ρl is

$$\rho l \frac{d^2 y}{dt^2} = p - \frac{T y l}{\alpha(l-\alpha)}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (1), (2), and (3) we obtain

$$\left. \begin{aligned} \frac{d^4 \xi}{dt^4} + \frac{T}{\alpha(l-\alpha)\rho} \cdot \frac{d^2 \xi}{dt^2} + (\mu K_1 + \mu T/M\rho\alpha(l-\alpha))\xi &= 0, \\ K_1 &= \frac{1}{l\rho} + \frac{1}{M}. \end{aligned} \right\} \quad (4)$$

Assuming $\xi = l^{st}$, we get

$$s^2 = -(\beta \pm \sqrt{\beta^2 - \gamma}),$$

giving four values of s .

$$2p = \mu K_1 + \frac{T}{\rho \alpha (l - \alpha)},$$

$$\gamma = \frac{T\mu}{M\rho\alpha(l - \alpha)}.$$

Two cases of interest immediately present themselves:—

- (1) When $\mu = \infty$: *i. e.* the hammer is hard, s comes out infinite, or the duration of impact is very small.
- (2) When $\mu = 0$, $s = \pm i T / \rho \alpha (l - \alpha)$: *i. e.* the duration of impact is the same as the period of vibration of the mass ρl .

Case II.

Length of the string finite.

The duration of impact is supposed to be small compared with the period of vibration of the string, which is therefore not appreciably displaced from its equilibrium position (Helmholtz).

$$y_0 = \xi.$$

Eliminating y_0 from (1) and (2) with the help of the above relation, we obtain

$$\frac{M}{\mu} \frac{d^2 p}{dt^2} = -p. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\therefore p = v_0 (\mu M)^{\frac{1}{2}} \sin \{ \mu / M \}^{\frac{1}{2}} t,$$

where v_0 represents the velocity with which the hammer impinges upon the string. The law of pressure is the same as supposed by Helmholtz. The duration of impact is given by $\pi (M / \mu)^{\frac{1}{2}}$; it depends upon the mass of the hammer and the strength of the spring, but it is independent of the striking distance and the velocity of impact.

Case III.

Length of the string infinite, and it is displaced during impact.

Now

$$p = \mu \xi = 2T \frac{dy}{dx} = \frac{2T}{C} \frac{dy}{dt} \text{ approx. } . \quad . \quad . \quad (6)$$

Eliminating y between (1) and (6), we obtain

$$\frac{M}{\mu} \frac{d^2 p}{dt^2} + \frac{MC}{2T} \frac{dp}{dt} + p = 0, \quad . \quad . \quad . \quad . \quad (7)$$

$$p = \left\{ \frac{\mu}{M} - \frac{1}{4} \left(\frac{\mu C}{2T} \right)^2 \right\}^{\frac{1}{2}} \sin \left\{ \left\{ \frac{\mu}{M} - \frac{1}{4} \left(\frac{\mu C}{2T} \right)^2 \right\}^{\frac{1}{2}} t \right\} \quad . \quad . \quad (8)$$

if $\frac{16T\rho}{\mu} > M$, *i.e.* if the hammer is elastic.

The duration of impact in this case is given by

$$\pi \left\{ \frac{\mu}{M} - \frac{1}{4} \left(\frac{\mu C}{2T} \right)^2 \right\}^{-\frac{1}{2}}.$$

We observe that the duration of impact for an elastic hammer is finite for infinite length of the string. It is independent of the velocity of the impact, depending mainly upon the mass and elasticity of the hammer. As μ increases, the duration of impact increases, and when μ is large, equation (7) degenerates into

$$\frac{MC}{2T} \frac{dp}{dt} + p = 0, \quad . \quad . \quad . \quad . \quad (9)$$

$$p = Ae^{-\frac{2T}{MC}t}, \quad . \quad . \quad . \quad . \quad (10)$$

A being a constant to be determined from the initial conditions. Thus we see from (10) that when μ is infinite, the duration of impact is also infinite for a very long string, in agreement with Kaufmann.

Case IV.

Length of the string finite; it is displaced during impact, and α , the striking distance, is small.

Further, we suppose that the portion of the string between the point struck and the nearer end follows the motion of the hammer. This means that the wave takes little time to reach and get reflected from the nearer extremity. The other wave travels onward towards the longer side and reaches the end on the longer side, when the impact is supposed to be over. The former assumption no doubt overestimates the kinetic energy, but the actual facts cannot be far from this. p is then given approximately by

$$p = \mu \xi = T \left\{ \frac{y}{\alpha} + \frac{1}{C} \frac{dy}{dt} \right\}. \quad . \quad . \quad . \quad . \quad (11)$$

Eliminating p from (1) and (11), we obtain to a first approximation

$$M_1 \left(1 + \frac{T}{\mu \alpha} \right) \frac{d^2 y}{dt^2} + \frac{T}{C} \frac{dy}{dt} + \frac{T y}{\alpha} = 0, \quad \dots \quad (12)$$

where

$$M_1 = M + \rho \alpha / 3.$$

$$\therefore y = A e^{-\frac{K}{2} t} \sin qt, \quad \dots \quad (13)$$

$$\left. \begin{aligned} q^2 &= \frac{T}{\alpha M_1} \left(\frac{\mu \alpha}{T + \mu \alpha} \right) - \frac{1}{4} K^2, \\ K &= \frac{T}{M_1 C} \left(\frac{\mu \alpha}{\mu \alpha + T} \right). \end{aligned} \right\} \quad \dots \quad (14)$$

If v_0 is the initial velocity of the hammer at the beginning of impact, A is found to be given by

$$A = \frac{v_0}{q} \left\{ 1 + \frac{T}{\mu} \left(\frac{1}{\alpha} - \frac{K}{2C} \right) \right\}^{-1} \quad \dots \quad (15)$$

From (13) and (15) we find in the case of hard hammer, i. e. $\mu = \infty$,

$$\left. \begin{aligned} q_1^2 &= \frac{T}{\alpha M_1} - \frac{1}{4} \left(\frac{T}{M_1 C} \right)^2, \\ A_1 &= \frac{v_0}{q_1}. \end{aligned} \right\} \quad \dots \quad (16)$$

The duration of impact in the case of an elastic hammer is given by π/q (approximately); it depends upon the mass of the hammer, its elasticity, and the striking distance α .

From a comparison with q_1 in (16), we observe that the effect of elasticity is to prolong the duration of impact. When the hammer is very soft, μ is very small. The duration of impact becomes infinite, as is also evident on general grounds. The duration of impact increases as α increases. It is, as before, independent of the initial velocity of impinge of the hammer. By comparing (14) and (15), we find that the duration of impact of an elastic hammer when the string is appreciably displaced from its equilibrium position during impact, is greater than that when there is no appreciable displacement of the string.

The amplitude of motion produced in the string is proportional to the initial velocity of impinge, and increases with α and the duration of impact. When μ is very small, the amplitude of motion is also very small; but when μ is

infinite it is finite, as shown by (16). Further, we observe that the amplitude in the case of an elastic hammer is greater than that due to a hard hammer.

III. *Experiments.*

The hammers used in the piano in the upper octaves are harder and lighter than those used in the middle ranges. The distance of the striking-point is almost the same for all strings; it varies within one-ninth to one-seventh of the length of the string in different types of piano. Hence, in order to test the above formulæ, a sonometer was fixed almost vertically and the hammer was placed near the lower extremity of the sonometer, so that the distance of the point struck from the lower bridge was small. By an arrangement of a lever the hammer was made to strike in a vertical plane—that is, parallel to the plane containing the hammer and the string. The lever arrangement was the same as is used in the piano. The wires and hammers used in the experiments were borrowed from a local piano-tuner, and those were selected which are used in the middle range where the hammer is elastic and heavy. The point of impact was photographed by a sliding plate which recorded the approach and retreat of the hammer. (Care was taken that the hammer did not strike the string a second time, by an arrangement of a catch.) The time was simultaneously obtained from the trace of a style attached to a tuning-fork. In Pl. VIII. figs. 1, 2, 3, 4, and 5 show in ascending order the distance of the point struck. The approach and retreat of the hammer are shown by the parabolic shadow, and the vibration of the point struck and the record of the style are also seen. It is evident from these figures that the duration of impact and the amplitude of the resulting motion increase with increasing distance of the striking-point. These facts are in agreement with those deduced from (14) and (15). Formula (14) can be written in the form:

$$\frac{\phi}{\theta} = \pi \sqrt{\left[\frac{M_0}{M_1} \left\{ \frac{4l}{\alpha \left(1 + \frac{T}{\mu \alpha} \right)} - \frac{M_0}{M_1} \frac{1}{\left(1 + \frac{T}{\mu \alpha} \right)^2} \right\} \right]^{\frac{1}{2}}}.$$

ϕ = Duration of impact.

θ = Period of vibration of the string.

Table I. gives a record of the values at different distances α found experimentally and calculated from (14).

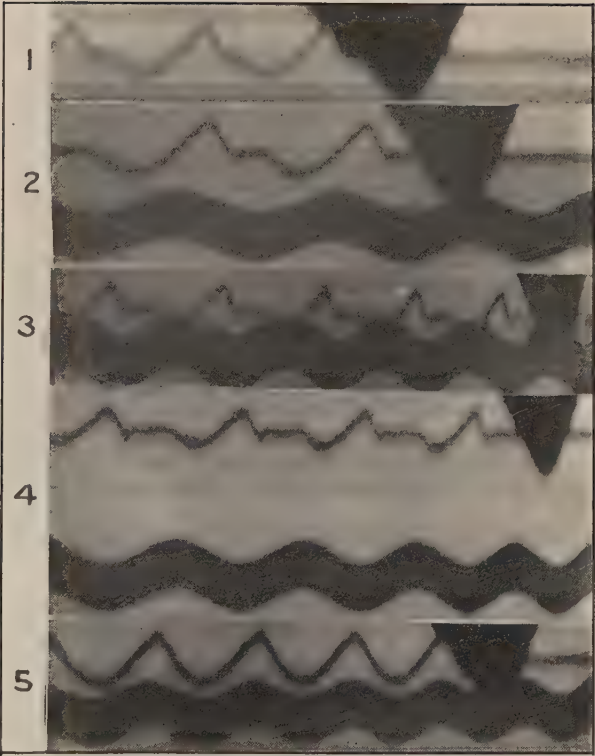


TABLE I.

 $\rho = .0589$ gram per cm. $M_0 = 5.61$ grams. $M = 7.3$ grams (mass of head plus one-third mass of stem). $\theta = \frac{1}{100}$ second.

Length of the string 95.4 cm.

 $T/\mu = 2.64$.

No.	α .	ϕ/θ obs.	ϕ/θ calc. (14).	ϕ/θ Kaufmann.
	cm.			
1 ...	8.2	.607	.604	.52
2 ...	9.8	.64	.65	.60
3 ...	11.5	.704	.706	.63
4 ...	14.3	.80	.79	.75
5 ...	17.7	1.0	.87	.84
6 ...	18.1	1.06	.90	.86

From the table it is clear that the values observed and those calculated from (14) agree fairly, while Kaufmann's formula gives more divergent values than experimental errors justify. Further, it is also confirmed that the duration of impact does not depend upon the velocity of impact, to which no fixed value was imposed during experimentation.

Hence it is concluded that to determine the law of pressure between the hammer and the string, its elasticity must be taken into consideration. In the higher scale of the piano, where the hammer is hard and light, Kaufmann's theory of inelastic impact may hold good. It is also found that the calculated values of ϕ/θ differ from those found experimentally with increasing striking distance. They are less than those observed. This regular decrease in the calculated values points out that our assumption, namely that the portion of the string between the striking-point and the nearer extremity follows the motion of the hammer, does not hold true when α is large, appreciable time being taken by the wave to reach the nearer end and return.

Conclusion.

Since in practice the striking distance is about one-ninth the length of the string, it is thought that the above theory

of the elastic hammer will be of practical value. It is more general than Kaufmann's, and includes his as a special case. The authors hope to calculate the intensity of partials on the basis of the law of pressure given by them, and compare it with experiments and verify formula (15) with different values of v_0 on a future occasion.

Physics Laboratory, Allahabad University,
Allahabad, India,
December 14th, 1922.

**CX. *The Relation between Uranium and Radium.*—
Part VIII. *The Period of Ionium and the Ionium-
Thorium Ratio in Colorado Carnotite and Joachimsthal
Pitchblende.* By FREDERICK SODDY, M.A., F.R.S., and
Miss ADA F. R. HITCHINS, B.Sc.***

IN the course of a study of the ionium-thorium ratio in minerals, occasion has been taken to redetermine the period of ionium from the rate of growth of radium in the various uranium preparations purified from 18 to 15 years ago, and the new results may first be recorded. It may be recalled that, as the result of the last examination in 1919 (Phil. Mag. [6] xxxviii. p. 483 (1919)), it was found that the rate of growth of radium was proceeding regularly in accordance with the square of the time, with the period of 237,500,000 years as the product of the periods of average life of ionium and radium, which led to the adoption of the period of 100,000 years for that of ionium. The new results are set forth in the same tabular form as in the paper cited. The value of the average life of radium is retained at 2375 years for the sake of uniformity, and the average life of ionium, $1/\lambda_2$, is calculated from the formula

$$1/\lambda_2 = \frac{1}{2}(U/Ra)(3.4 \times 10^{-7}/2375)T^2,$$

where Ra grams of radium are produced from U grams of uranium in T years.

Two sets of determinations have been made, in August and December 1923, of the amount of radium in the four uranium preparations. The electroscope was first calibrated with a number of radium standards, made from a radium barium chloride of known small radium content, one of which was new and the others those previously employed.

* Communicated by the Authors.

Calibration of the electroscope.

Standard g. BaCl ₂ .	Ra ($\times 10^{-12}$ g.).	Divisions per minute.	Constant 10^{-12} g. of radium giving 1 d.p.m.
0.0454	29.6	1.86	15.91
0.2140 (new)	139.5	8.78	15.89
0.2625	171.1	10.91	15.68
0.3981	259.6	16.47	15.76
0.5683	370.5	24.16	15.34
0.7869	512.9	33.02	15.53
Mean			<u>15.69</u>

Before the measurements were repeated in December the electroscope was again calibrated :

0.2704	176.3	11.21	15.73
0.5683	370.5	23.91	15.50
0.7869	512.9	33.36	15.37

Since no significant change had occurred the constant 15.69 was used throughout.

The results are given in the following table:—

Preparation.	Grams Uranium.	Time (years).	(Time) ² .	Radium ($\times 10^{-12}$ g.).	Initial Radium.	Growth of Radium.	Growth of Radium \div (Time) ² .	Ditto. Mean Value.	Period of Ionium (years).
I.	255	17.81	317.2	71.7	16	55.7	0.176	0.175	104,300
		18.15	329.4	73.3		57.3	0.174		
II.	278	17.00	289.0	72.0	18	54.0	0.187	0.189	105,300
		17.34	300.7	75.5		57.5	0.191		
III.	408	16.68	278.2	75.5	4	71.5	0.257	0.2585	113,000
		17.02	289.7	79.4		75.4	0.260		
V.	3000	14.21	201.9	434.5	40	394.5	1.954	1.958	109,700
		14.54	211.4	454.7		414.7	1.962		
Simple Mean									108,000

For the first time since the start, the older preparations I. and II., in which the initial radium exceeded one part

in 10^{14} of uranium, are sufficiently far advanced to give quantitative evidence, and this is in agreement with that given by the two later preparations. The new mean result is 8 per cent. higher than that arrived at in 1919, and this is somewhat outside the probable degree of accuracy (5 per cent.) then estimated. This would of course be explained by the presence of initial ionium, but in view of the isotopism of uranium X and the known behaviour of the latter under the methods of purification employed, it is practically impossible to believe that sufficient ionium to account for the result was initially present. It seems more natural to ascribe it to the errors of measurement being for some reason larger than was supposed and to round off the value to be adopted for the average life of ionium to

$$\underline{1.1 \times 10^5 \text{ years.}}$$

Determination of the Ionium-Thorium ratio in Minerals.

Provided sufficient can be obtained to weigh, since ionium and thorium are isotopes, the ionium-thorium ratio in a mineral may be obtained, when the period of ionium is known, by separating and purifying the ionium-thorium mixture, with precautions against the adventitious introduction of thorium, and determining the rate of growth of radium in a preparation containing a known weight. Ordinarily, a small uncertainty is introduced, especially in intensely active preparations, by the time of accumulation of the emanation between measurements, during which time the radium itself is changing in quantity,—in an active preparation perceptibly. For a weak preparation in which the emanation may be allowed to attain “transient” equilibrium, the correction is exactly and readily made by subtracting from the total time of accumulation of the radium the period of average life of the emanation. But the correction may be made exactly and readily for any period of accumulation.

The first differential equation of the disintegration series involved, in which I_0 and R_a refer to (atomic) quantities of ionium and radium, λ_2 and λ_3 being their radioactive constants,

$$d(R_a)/dt = \lambda_2 I_0 - \lambda_3 R_a$$

gives, since the second term is negligible relatively to the first,

$$R_a - R_{a_0} = \lambda_2 I_0 t,$$

where R_{a_0} is the initial quantity of radium present and t is the time.

The second differential equation of the series

$$d(\text{Rn})/dt = \lambda_3 \text{Ra} - \lambda_4 \text{Rn},$$

where λ_4 is the constant of radon (radium emanation), gives, by substituting the above value for Ra,

$$\text{Rn} = (\lambda_2 \lambda_3 / \lambda_4) \text{Io} (t - 1/\lambda_4) + \lambda_3 \text{Ra}_0 + \text{C} e^{-\lambda_4 t},$$

where C is the integration constant.

Solving for the conditions that $\text{Rn} = 0$, when $t = t_0$, that is, that the time from purification to the time of sealing up the preparation is t_0 , and that the time of accumulation of radon is τ , the total time from purification to measurement being $t_0 + \tau$, gives

$$\lambda_4 \text{Rn} / \lambda_3 (1 - e^{-\lambda_4 \tau}) = \lambda_2 \text{Io} \{t_0 - 1/\lambda_4 + \tau / (1 - e^{-\lambda_4 \tau})\} + \text{Ra}_0.$$

The quantity on the left-hand side of the equation is the quantity of radium measured in the usual way in the emanation method of estimation, being the emanation corrected to equilibrium accumulation and expressed as the quantity of radium with which the corrected quantity of emanation is in equilibrium. [For if a quantity of radium Ra produces a quantity of radon Rn in time of accumulation τ , and Rn_e is the equilibrium quantity,

$$\text{Rn} = \text{Rn}_e (1 - e^{-\lambda_4 \tau}) \quad \text{and} \quad \lambda_4 \text{Rn}_e = \lambda_3 \text{Ra}.$$

Hence
$$\text{Ra} = \lambda_4 \text{Rn} / \lambda_3 (1 - e^{-\lambda_4 \tau}).]$$

Substituting and converting atomic weights to weights gives

$$\text{Ra} - \text{Ra}_0 = \lambda_2 \text{Io} \{t_0 - 1/\lambda_4 + \tau / (1 - e^{-\lambda_4 \tau})\} (226/230). \quad (1)$$

We may designate the expression within {} "the effective time" θ , where

$$\theta = t_0 - 1/\lambda_4 + \tau / (1 - e^{-\lambda_4 \tau}). \quad (2)$$

Putting in the values $1.1 \times 10^5 \times 365$ for $1/\lambda_2$ (days), and 5.555 for $1/\lambda_4$ (days), gives

$$\text{Ra} - \text{Ra}_0 = 2.45 \times 10^{-8} \text{Io} \theta, \quad (3)$$

and
$$\text{Io} = 4.08 \times 10^7 (\text{Ra} - \text{Ra}_0) / \theta, \quad (4)$$

$$\theta = t_0 - 5.555 + \tau / (1 - e^{-\lambda_4 \tau}).$$

These expressions allow the quantity of ionium to be rapidly and exactly estimated without the necessity of

1152 Prof. F. Soddy and Miss A. F. R. Hitchins on the waiting for "transient" equilibrium, that is for $(1 - e^{-\lambda_1 \tau})$ to become unity.

American Carnotite.

The first mineral to be examined for the ionium-thorium ratio by the above method was American carnotite. A specimen containing some 60 grams of ionium-thorium oxides was sent to us for examination by Mr. W. F. Bleecker, of Boulder, Colorado, U.S.A. It had been separated in his radium factory from hundreds of tons of representative carnotite ore, and from measurements of the rate of growth of radium he had already made it was estimated to contain about 4 per cent. of ionium. In connexion with it he stated that "so far as it is humanly possible for me to tell, no thorium was added to the ionium at any time during its preparation."

A small part of the material was purified from radium by successive precipitations of its solution by ammonia, and then made up to a known weight of solution. One part of this was evaporated and ignited to determine the IoO_2 , ThO_2 content, and the other used for the radium method of estimating the ionium, the weight of IoO_2 , ThO_2 in this part being 75.8 milligrams. The initial radium determined by graphing the results and extrapolating to zero time was 4×10^{-11} gram, and this has been subtracted in the results shown in the following table:—

No. of estimation.	$\text{Ra} - \text{Ra}_0$ ($\times 10^{-11}$ g.).	t_0 (days).	τ (days).	Effective Time θ (days).	Ionium (mg.).
1	3.6	0	1	0.52	2.81
2	9.3	1	1	1.52	2.49
3	17.0	2	0.96	2.51	2.78
4	26.1	3	2	4.06	2.62
5	48.4	5	4	7.24	2.72
6	96.5	9	8	13.93	2.83
7	174.9	17	12	25.01	2.85
8	243.3	29	9	34.67	2.87
Mean					2.75

The quantity of ionium as determined by the mean is 2.75, and by the mean of three final results, which may be regarded as the more trustworthy, 2.85 mg. Accepting the latter figure, the IoO_2 content is 3.23 mg., the percentage of IoO_2 in the oxides 4.26, and the ratio $\text{Io} : \text{Th}$ is 1 to 23.

The Mesothorium-Radium ratio of Radium from Carnotite.—

Assuming that no contamination with thorium has occurred during manufacture, it follows that in carnotite there are 23 grams of thorium per gram of ionium or 1.05 kilograms of thorium per gram of radium. McCoy and Henderson (J. Amer. Chem. Soc. xl. p. 1316, 1918) give 19 kilograms as the quantity of thorium in equilibrium with the "milligram of mesothorium," *i.e.* the quantity of mesothorium which gives a γ -radiation measured through 2 mm. of lead and 1.32 mm. of brass equal to that of a milligram of radium in equilibrium with its γ -ray producing products. Hence per gram of radium separated from American carnotite there must be 0.055 "milligram of mesothorium," or the mesothorium contributes initially one part in 1800 of the γ -radiation. The same authors show that the "milligram of mesothorium," when in equilibrium with radiothorium, gives a total γ -radiation equal to 2.7 mg. of radium. So that as much as 3 parts per 2000 of the γ -radiation of carnotite radium may, as a maximum, be due to thorium products. This is below the error of present methods of γ -ray measurements, and there does not seem to be any valid reason against the use of carnotite radium for standards of measurements.

Joachimsthal Pitchblende.

Being in possession, through the courtesy of the Mines Department of the Government of Czecho-Slovakia, of a set of products from the State radium factory at St. Joachimsthal (now Jachymov, Czecho-Slovakia), upon which much chemical work has been done, it was thought to be of interest to determine the ionium-thorium ratio for this mineral. The method of attack in separating the ionium and actinium from these products was the one now well-established. The behaviour of the ionium and the actinium during the chemical treatments was followed by means of isotopes as indicators, uranium X being used for ionium and mesothorium-2 for actinium, β -ray methods of measurements being employed. Our labours in concentrating the constituents on a large scale were much lightened by a very careful and thorough preliminary examination that had been made of the first product dealt with, by Mr. M. Garrett, who devoted a year to this work.

The material was the "Oxalsäureniederschlag der seltenen Erden," obtained from the first solutions, extracted from the insoluble radium residues of Joachimsthal pitchblende by successive treatments with alkali and acid, by precipitation with oxalic acid. The thorium-ionium ultimately separated, though purified repeatedly by the peroxide method until the

filtrate failed to give a precipitate on long standing with ammonia, was not completely pure. For on solution in nitric acid temporary discoloration and slight decomposition of the acid occurred. However, this behaviour is observed with thorium preparations containing less than 1 per cent. of impurity, and it is probable that the purity of the material finally separated exceeded 99 per cent. It weighed, as oxide, 44.3 milligrams, and a portion corresponding with 8.9 milligrams of oxide was used for the radium method of estimating the ionium.

No. of estimation.	$Ra - Ra_0$ ($\times 10^{-11}$ g.)	t_0 (days).	τ (days).	Effective Time θ (days).	Ionium (mg.).
1	16.2	0.06	2.75	1.55	4.25
2	45.4	2.8	3	4.44	4.16
3	81.6	5.8	4	8.04	4.13
4	128.7	9.8	5	12.67	4.13
5	163.8	14.8	3	16.44	4.06
6	499.6	49	1	49.52	4.12
Mean					4.14

The mean value 4.14 mg. of ionium corresponds with 4.7 mg. of oxide. The proportion of ionium oxide in the preparation is 53 per cent., and the ionium-thorium ratio is 1:0.89.

Now this was a very surprising result, because hitherto this ratio has been considered to be 3:7, whereas we find actually more ionium than thorium. In 1908, the radium residues from 30,000 kg. of Joachimsthal pitchblende of average uranium content 45 per cent. had been worked up by Haitinger and Ulrich in the chemical factory of the Austrian Gasglühlicht und Elektrizitätsgesellschaft. Out of the rare-earth fraction, A. v. Welsbach, in 1910, separated a few grams of pure ionium-thorium oxide, upon which most of the subsequent work has been performed. From the atomic weight, 231.51, obtained by Hönigschmid and Horowitz in 1916, regarded as a mean between that of thorium 232.12 and of ionium 230, the ionium-thorium ratio works out as 3:7. Stefan Meyer, comparing the α -radiation of this preparation with that of radium, found, allowing for the slight difference in the ranges of the α -particles, that one gram of the former emitted as many α -particles as 5.07 milligrams of radium, and from this calculated the period of ionium as 1.4×10^5 years, when that of radium is 2400 years*.

If we recalculate our result by formula (4) substituting 1.4×10^5 for 1.1×10^5 for $1/\lambda_2$, there would be 5.27 mg. Io, = 6.0 mg. of IoO_2 , in the 8.9 mg. of oxides, or 67.4 per cent.,

* Compare Ann. Rep. Chem. Soc. 1916, p. 251; Stefan Meyer, *Sitzungsber. k. Akad. Wiss. Wien*, II. a, cxxv. p. 191 (1916).

and the ionium-thorium ratio would be practically 2 : 1. Or, conversely, if we calculate the period of ionium from our result for the rate of formation of radium, assuming that the material is 30 per cent. ionium, the result is 7×10^4 years. So that it did not seem possible that the ionium-thorium ratio in Joachimsthal pitchblende could be as low as that hitherto accepted.

In order to check the result, another of the products from the State factory at St. Joachimsthal was worked up for ionium and actinium. This product was the "Sodaniederschlag." So far as we are aware it has not previously been chemically examined, although, from our present knowledge of the chemistry of the radio-elements, it is to be expected to contain actinium and ionium in notable amount. It is derived, not from the radium residue, which is the part of the mineral insoluble in sulphuric acid, but from the uranium solution, during purification of the uranium, as a precipitate when the solution is treated with sodium carbonate, the uranium remaining dissolved.

The ionium-thorium oxides separated from these residues weighed 19.7 mg., and a part equivalent to 3.9 milligrams was used for the radium estimation of ionium. Both in this and the preceding case no initial radium was present.

No. of estimation.	Ra—Ra ₀ ($\times 10^{-11}$ g.).	t_0 (days).	τ (days).	Effective Time θ (days).	Ionium (mg.).
1	10.5	0.17	3.75	2.26	1.89
2	24.4	4	3	5.64	1.77
3	37.8	7	3	8.64	1.78
4	51.6	10	4	12.24	1.72
5	147.6	33	1	33.52	1.79
Mean					1.79

1.79 mg. of ionium correspond with 2.05 mg. of oxide and the proportion of ionium oxide in the preparation is 52.5 per cent. The ionium-thorium ratio is as 1 : 0.9, substantially the same as before. Thus the very high proportion of ionium in the material is confirmed for another and different product of the same material.

The Mesothorium-Radium ratio for Radium from Joachimsthal pitchblende.—Accepting the ionium-thorium ratio as 1 : 0.9, and the period of ionium as 1.1×10^5 years, per gram of radium in the pitchblende there are 46 grams of ionium and 41 grams of thorium, a total of 87 grams of thorium isotopes. Or, per 1000 kg. of uranium ($= 0.34$ g. radium) there are respectively 15.6, and 14, with a total of 29.6 grams. The "milligram of mesothorium" is initially present in 450 grams of the radium from this mineral, so that one part

in 450,000 of the γ -radiation is due to mesothorium, or, including the radiothorium, 6 parts in a million, as a maximum, is due to thorium products. This is, of course, of no significance as regards the constancy of the γ -rays, though it may be easily put into evidence by chemical methods ('Nature,' xci. p. 934, 1913).

The recent results of Meyer and Ulrich.

Since the work described was completed, our attention has been directed to a recent paper by Meyer and Ulrich (*Sitzungsber. Akad. Wiss. Wien, II. a*, cxxxii. p. 279, 1923), in which some of our results have been anticipated. They have examined the ionium-thorium content of Joachimsthal pitchblende to ascertain whether specimens from different quantities of the material had the same composition, particularly from the point of view of a suggestion of G. Kirsch*, that an isotope of uranium of atomic weight 236, and period about 10^8 years, may be the first member of the thorium disintegration series, and that thorium may be the product of this in an α -ray change. In this event, if Joachimsthal pitchblende were a mineral of recent formation, different specimens of different age might show different ionium-thorium content.

Meyer and Ulrich have prepared two new thorium-ionium preparations from the rare-earths, separated in various stages of the manufacturing process, from the mineral worked up in the State factory at St. Joachimsthal in 1912 and 1913. The first of these only is as yet sufficiently free from the isotopic radioactinium to allow of an exact α ray comparison with the preparation for which the atomic weight 231.51 was found. The new preparation was found to be more active in the ratio 100 : 60, and the authors conclude that if the latter is 30 per cent. ionium, the new preparation must consist of very nearly equal quantities of ionium and thorium, and should possess an atomic weight of 231.1. This result is thus precisely what we have found, although the methods of estimating the ionium are totally different, and, as has been shown, not entirely consistent. The authors conclude that their results make it certain that the ionium-thorium mixture from different samples of Joachimsthal pitchblende must vary between 30 and 50 per cent., and they discuss this in connexion with the supposed parent of thorium.

In our view, however, the probable explanation is not so far to seek. Since we have found essentially the same 1 : 1 ionium-thorium ratio as Meyer and Ulrich, the natural conclusion is that the preparation separated by Welsbach, for

* G. Kirsch, *Sitzungsber. Akad. Wiss. Wien, II. a*, cxxxi. p. 551 (1922).

which a 3:7 ratio was found, is exceptional, and has probably been contaminated by extraneous thorium. It was worked up in 1908, before anything was known about isotopes, by Haitinger and Ulrich in a thorium factory, and the dust settling from the air, without any more direct contamination, might well account for the difference, even before the ten tons of raw material found its way into Welsbach's hands. The subsequent separation of the ionium occupied two years, and presumably, though the fact is not explicitly on record, this further working up took place also in a thorium factory. Indeed it must be considered remarkable that the contamination was not greater, and the 3:7 ratio must be rather regarded as eloquent of the great cleanliness and care with which this, then extraordinarily complicated and difficult operation, was conducted.

Summary.

1. Recent redeterminations of the average-life period of ionium, from the rate of growth of radium in four uranium preparations purified from 18 to 15 years ago, give for the mean value 1.08×10^5 years. This is 8 per cent. higher than the value arrived at in 1919, and it is suggested that the rounded value 1.1×10^5 years should be meantime adopted.

2. An exact method is described for the determination of ionium, assuming this period from the rate of growth of radium, and which is applicable to very active preparations. The "effective time" of accumulation, θ , is given by

$$\theta = t_0 - 5.555 + \tau / (1 - e^{-\lambda_4 \tau}),$$

where t_0 is the time from purification to sealing the preparation, τ the time of accumulation of emanation, and λ_4 the radioactive constant of the latter, in days. Then

$$I_0 = 4.08 \times 10^7 (Ra - Ra_0) / \theta,$$

where $Ra - Ra_0$ is the radium produced by the quantity I_0 of ionium (grms.).

3. An examination of an ionium-thorium mixture separated from many hundreds of tons of representative American carnotite, gave the ionium-thorium ratio as 1:23. Radium prepared from this source will have as a maximum 3 parts per 2000 of the γ -radiation contributed by mesothorium and radiothorium.

4. Two specimens of ionium-thorium have been separated from different products of the working up of the Joachimsthal pitchblende, the first from the oxalic acid precipitate, containing rare-earths, separated from the insoluble radium

residues, and the second from the sodium carbonate precipitate from the solution of the mineral in sulphuric acid, in the uranium purification. Both specimens showed that the ionium-thorium ratio was 1:0.9, the ionium actually predominating.

5. Results recently published by Meyer and Ulrich on a thorium-ionium preparation separated from the rare-earths in the material worked up in St. Joachimsthal in 1912 and 1913, give an α -radiation greater than that prepared by Haitinger and Ulrich and A. v. Welsbach in 1908-1910, for which the atomic weight 231.51 and the ionium-thorium ratio 3:7 have been accepted. This is in agreement with our result that the quantities of ionium and thorium in the mineral are nearly equal. We suggest that the older preparation was contaminated through having been worked up in a thorium factory.

We wish to express our acknowledgements to Mr. Bleecker of Boulder, Colorado, and to the Department of Mines of Czecho-Slovakia, for the materials examined, and to Mr. M. Garrett, Rhodes Scholar of Exeter College, for valuable assistance in chemical investigations preliminary to this work.

Old Chemistry Department,
University Museum, Oxford,
April 19th, 1924.

CXI. *Note on the Relativity Dynamics of a Particle.*

By P. A. M. DIRAC, *St. John's College, Cambridge* *.

A VELOCITY corresponds to a direction in four-dimensional space-time, and is determined by the ratios of the components of a vector drawn in that direction. The velocity of matter may be defined in two ways. If we can identify and follow the motion of a particular particle or other discontinuity, the direction of its world-line gives us the "kinematical velocity." Alternatively, if we measure the three components of momentum-density and the mass-density at any point, then in certain cases (which include that of an isolated particle) the direction determined by the four quantities is invariant, and it then gives the "dynamical velocity." In Eddington's derivation of the gravitational equations †, mass and momentum are defined through their property of conservation without reference to the conception

* Communicated by Prof. A. S. Eddington, F.R.S.

† 'The Mathematical Theory of Relativity,' Ch. iv.

of kinematical velocity, so that the identity of the two definitions is not immediately obvious. Eddington has proved the identity of the definitions for an isolated particle when it has three perpendicular planes of symmetry*; but it can be done without any postulates concerning the form of the particle, by considering the conditions that hold at surfaces of discontinuity in a distribution of matter.

The simplest case in which kinematical velocity has a meaning is that of a (three-dimensional) surface of discontinuity separating a region containing a continuous distribution of matter from a region containing none; in that case the normal component of the kinematical velocity of the matter at the surface is zero in any system of coordinates for which the equation of the surface does not involve x_4 . We choose our system of coordinates such that the equation of the surface is $x_1 = \text{constant}$. Since the divergence of the material energy-tensor vanishes, we have in the usual notation

$$\frac{\partial T^{\mu\nu}}{\partial x_\nu} + \{\alpha\nu, \mu\}T^{\alpha\nu} + \{\beta\nu, \nu\}T^{\mu\beta} = 0,$$

which may be written

$$\frac{\partial T^{\mu 1}}{\partial x_1} = -\frac{\partial T^{\mu 2}}{\partial x_2} - \frac{\partial T^{\mu 3}}{\partial x_3} - \frac{\partial T^{\mu 4}}{\partial x_4} - \{\alpha\nu, \mu\}T^{\alpha\nu} - \{\beta\nu, \nu\}T^{\mu\beta}.$$

The terms on the right are all finite at the boundary, however rapid the transition from finite density to zero density may be. Hence $\frac{\partial T^{\mu 1}}{\partial x_1}$ is also finite at the boundary, so that $T^{\mu 1}$ is continuous, and therefore vanishes just inside the region containing the matter. So the normal component of the dynamical velocity of the matter at the boundary vanishes, in agreement with all that is known concerning the kinematical velocity.

We now consider a particle in free space, whose equations may be taken to be $x_1 = \text{constant}$, $x_2 = \text{constant}$, $x_3 = \text{constant}$, so that its kinematical velocity vanishes in these coordinates. Then $T^{\mu 1}$, $T^{\mu 2}$, and $T^{\mu 3}$ are continuous at the surface of the tube which is occupied by the particle, and therefore vanish inside it. We are here assuming that the differences in $T^{\mu\nu}$ at different points of a section of the tube are negligible, this being what is meant by a simple particle. So all the components of $T^{\mu\nu}$ vanish except T^{44} , and the dynamical velocity is zero. Hence the dynamical velocity vector has the same direction as the kinematical velocity vector, viz. along x_4 .

* *Loc. cit.* § 56.

CXII. *Notices respecting New Books.*

An Introduction to the Theory of Optics. By Sir ARTHUR SCHUSTER, Sc.D., F.R.S., &c. Third Edition, revised and enlarged by the Author and J.W. NICHOLSON, M.A., D.Sc., F.R.S. (Pp. xv + 397, with 188 figures.) (London : Edward Arnold & Co., 1924. Price 18s. net.)

A Treatise on Light. By R. A. HOUSTOUN, M.A., Ph.D., D.Sc., Lecturer on Physical Optics in the University of Glasgow. Fourth Edition. (Pp. x + 486, with 334 figures and 2 plates.) (London : Longmans, Green & Co., 1924. Price 12s. 6d. net.)

BOTH of these well-known works on optics have passed into new editions. The previous editions of Schuster's work appeared in 1904 and 1909 respectively, whilst those of Houston's appeared in 1915, 1921, and 1923. The additions and changes in the new editions are in both instances comparatively slight and therefore detailed comment is not necessary.

The most important addition to the 'Theory of Optics' consists of the addition of two new chapters entitled "Emission Spectra and the Quantum Theory" and "Dynamical Theory of Spectra" respectively. These are presumably by Prof. Nicholson, an acknowledged authority on the subjects in question. The volume will long continue to remain the best introduction from the physical standpoint to the theory of optics. On account of its originality of treatment and the clearness and precision of the English it should be carefully read by every serious student of the subject. A word of praise is due to the publishers for the excellence of the printing.

The new matter in the latest edition of Houston's work consists of a very brief chapter on recent advances, a new index, and two coloured plates. The scope of this volume, which is intended for students who have completed a first year's course in physics and are proceeding further with the study of light, is a very comprehensive one. It is divided into four parts, dealing respectively with Geometrical Optics, Physical Optics, Spectroscopy and Photometry, and the Mathematical Theory. A student who has mastered this book will be well equipped for the more detailed study of any special portion of the subject. The author's considerable experience in teaching has enabled him to smooth out the difficulties which the student is likely to encounter and to present the subject in the simplest manner. The chapter dealing with systems of lenses may be mentioned as an example of simplicity of treatment. That a fourth edition is now required is proof that the volume has met a real need. The paper and printing could be improved upon but, on the other hand, the price has been kept to a reasonable figure, an important consideration for a volume which is designed as a student's textbook.

INDEX TO VOL. XLVII.

- A**BSORPTION coefficients, on the application of statistical equilibrium to the determination of, 209.
- Acidity, on the origin of, 1021.
- Addenbrooke (G. L.) on the connexion between the dielectric and other physical properties of the non-metallic elements, 945.
- Adiabatic invariance, on some dynamical illustrations of, 754.
- Adjustment of observations, on the, 816.
- Aeroplane monocoque fuselages, on the stresses in, 319.
- coolers, on the heat dissipation from, 324.
- Æsculin solutions, on the fluorescence of, 648.
- Ahmed (M. S.) on the secondary failure of thin tubes, 319.
- Air, on the resistance of, to falling spheres, 173; on the emission from an incandescent platinum wire in, 689; on the entrainment of, by a jet of gas, 1028.
- Allen (Prof. F.) on the critical frequency of pulsation of tones, 50, 126, 941.
- Alpha particles, on the capture and loss of electrons by, 277.
- Aluminium; on the H-particles from, 500.
- Anderson (W.) on a consequence of the theory of M. L. de Broglie, 873.
- Angle of contact between paraffin wax and water, on the, 91.
- Anhyseretic qualities of iron and nickel, on the, 848.
- Anode rays, on accelerated, 385.
- Apparatus for purification of radium emanation, on an, 713.
- Appleton (Dr. E. V.) on the anomalous behaviour of a vibration galvanometer, 609.
- Arc lamp, on the potential gradient in the sodium potassium, 603.
- spectrum of phosphorus, on the, 874.
- Argon, on the kinetic energy of electrons from tungsten in, 458.
- Arsenic, on the crystal structure of metallic, 657.
- Ashworth (Dr. J. R.) on the anhyseretic qualities of iron and nickel, 848.
- Aston (Dr. F. W.) on the mass-spectra of chemical elements, 385.
- Atomic models, calculations of the potential energy for some, 992.
- Atoms, on the artificial disintegration of, 500; on the relation between refractivities and sizes of, 772.
- Aural fatigue, on the effect of, on the critical frequency of pulsation of tones, 126, 141.
- Auto-transformer, on the, 241, 1092.
- Ayres (Dr. T. L. R.) on ionization by collision in helium, 401.
- Backhurst (I.) on an all-metal high-vacuum pump system, 918, 1016.
- Bhargava (S.) on the elastic impact of a pianoforte hammer, 1141.
- Bailey (Prof. V. A.) on the motion of electrons in neon, 379.
- Balmer series of hydrogen, on the transitions involved in the, 826.
- Bamber (Miss A. E.) on the Avonian of the Western Mendips, 621.
- Barkla (Prof. C. G.) on corpuscular radiation excited by X-rays, 1.
- Barton (Prof. E. H.) on viscosities of liquids experimentally correlated to pendulum dampings, 495.
- Bell (H.) on the halogen hydrides, 549.

- Bending stresses in thin-walled tubes, on, 197.
- Beryllium, on the mass-spectrum of, 393.
- Bessel functions, tables of, 62, 736.
- Bohr (Prof. N.) on the quantum theory of radiation, 785.
- Books, new:—Bose's *Life Movements in Plants*, 250; Lamb's *Dynamics*, 250; McEwen's *The Properties of Matter*, 250; Bosler's *L'Évolution des Étoiles*, 251; Haas's *The New Physics*, 252; Graetz's *Recent Developments in Atomic Theory*, 252; Rice's *Relativity*, 471; Watson's *Treatise on the Theory of Bessel Functions*, 472; Clayton's *Introduction to the Study of Alternating Currents*, 473; Kramers and Holst's *The Atom and the Bohr Theory of its Structure*, 474; Whetham's *The Theory of Experimental Electricity*, 474; Lorentz's *Clerk Maxwell's Electromagnetic Theory*, 475; Mellor's *Comprehensive Treatise on Inorganic and Theoretical Chemistry*, 475; Mottelay's *Bibliographical History of Electricity and Magnetism*, 476; Worsnop and Flint's *Advanced Practical Physics for Students*, 477; Ricci and Levi-Civita's *Méthodes de Calcul Différentiel et leurs Applications*, 477; Campbell's *The Structure of the Atom*, 478; Freundlich's *The Theory of Relativity*, 781; Freundlich's *The Foundations of Einstein's Theory of Gravitation*, 781; Muir's *The Theory of Determinants in the Historical Order of Development*, 782; Nunn's *Relativity and Gravitation*, 782; Schuster's *Introduction to the Theory of Optics*, 1160; Houstoun's *Treatise on Light*, 1160.
- Boulton (Prof. W. S.) on a breccia-bed underlying Nechells, 621.
- Bradley (A. J.) on the crystal structure of metallic arsenic, 657.
- Breit (Dr. G.) on the polarization of resonance radiation, 832.
- Briggs (Dr. S. H. C.) on electrovalence and covalence, 702.
- Broglie (M. L. de) on a tentative theory of light quanta, 446.
- Browning (Dr. H. M.) on viscosities of liquids experimentally correlated to pendulum dampings, 495.
- Burton (Dr. E. F.) on the distribution of colloidal particles, 721.
- Cæsium, on the vapour pressure of, 32.
- Calcium, on the mass-spectrum of, 394.
- Calderwood (J.) on the increase of inertia of bodies vibrating in a liquid, 624.
- Calthrop (J. E.) on the relation between refractivities and sizes of atoms, 772.
- Campbell (Dr. N.) on ultimate rational units, 159; on dimensional analysis, 481; on the adjustment of observations, 816.
- Carbon, on soft X-rays from, 466.
- Carnotite, on the ionium-thorium ratio in, 1148.
- Case (J.) on bending stresses in thin-walled tubes, 197.
- Chromium, on the mass-spectrum of, 397.
- Cobalt, on the mass-spectrum of, 395.
- Colloidal particles, on the distribution of, 721.
- Coloured flames, on an automatic feeder for, 859.
- Comets, on a diffusion theory of, 724.
- Congdon (J. E.) on the kinetic energy of electrons emitted from hot tungsten, 458.
- Convective cooling in liquids, on, 972, 1057.
- Copper, on the corpuscular radiation from, 18; on the mass-spectrum of, 395.
- Core (A. F.) on the theory of solutions, 977.
- Corpuscular radiation excited by X-rays, on, 1.
- Coupled circuits, on valve-generated oscillations in, 625.
- Covalence, on electrovalence and, 702.
- Crystal structure of metallic arsenic, on the, 657.
- Currie (J. E.) on the distribution of colloidal particles, 721.
- Curvature invariant of space-time, on the, 907.

- Dallas (Miss A. E. M. M.) on the corpuscular radiation excited by X-rays, 1.
- Damped vibrations, on, 303.
- Davies (Dr. A. C.) on the emission of secondary electrons from metals under electronic bombardment, 245.
- Davis (A. H.) on convective cooling in liquids, 972, 1057.
- Davissou (Dr. C. J.) on the thermodynamics of thermionic emission, 544.
- De Sitter's world, geometrical note on, 930.
- Dielectric and other physical properties, on the connexion between, 945.
- Diffraction and the inertia principle, on, 452.
- image of two close luminous lines, on the, 1105.
- Diffusion theory of comets, on a, 724.
- Dimensional analysis, on, 481.
- Dirac (P. A. M.) on the relativity dynamics of a particle, 1158.
- Du Val (P.) on de Sitter's world, 930.
- Earth's charge, on the theory of the, 306.
- rotation, on the influence of the, on a top, 525.
- Einstein's theory, on the identical relations in, 584.
- Elastic stress, on certain types of, 679; on a simple solution of the general equations of, 965.
- Elasticity, on two-dimensional problems in, 1095.
- Electric current, on the effect of an, on the motion of mercury globules in dilute sulphuric acid, 802.
- moment and molecular structure, on, 530.
- properties of helium, on the, 737.
- Electrometer, on a new form of, 577; on the Lippmann capillary, 814.
- Electronic impacts, on statistical equilibrium and ionization by, 257.
- theory of valency, on the, 569, 1021.
- Electrons, on the emission of secondary, from metals, 245; on the capture and loss of, by alpha particles, 277; on the absence of ionization by, with speeds comparable with that of light, 306; on the free paths of, 347; on the motion of, in neon, 379; on the capture and loss of, by swift nuclei, 416; on the kinetic energy of, emitted from a tungsten filament, 458; on the emission of, from hot bodies, 561.
- Electrovalence and covalence, on, 702.
- Elements, on the characteristic frequency of, 84; on the mass-spectra of chemical, 385; on the connexion between dielectric and other physical properties of non-metallic, 945; on the complexity of the, 1121.
- Entrainment of air by a jet of gas, on the, 1048.
- Equation of state for fluids, on an, 431.
- Everett (Miss A.) on formulæ for oblique focal distance, 864.
- Falling spheres, on the resistance of air to, 173.
- Farquharson (D. J.) on the geology of southern Guernsey, 480.
- Ferguson (Dr. A.) on the angle of contact between paraffin wax and water, 91.
- Fisher (J. W.) on a supposed limitation of the second law of thermodynamics, 779.
- Flames, on an automatic feeder for coloured, 859.
- Fluids, on an equation of state for, 431; on the mean distance between molecules in, 671.
- Fluorescence of æsculin solutions, on the, 648.
- Flürscheim (Dr. B.) on the electronic theory of valency, 569.
- Focal distance, on formulæ for oblique, 864.
- Fowler (R. H.) on statistical equilibrium and the mechanism of ionization by electronic impacts, 257; on the capture and loss of electrons by swift nuclei, 416.
- Gallium, on the mass-spectrum of, 397.
- Galvanometer, on the anomalous behaviour of a vibration, 609.
- Gas, on the entrainment of air by a jet of, 1028.

- Gases, on the molecular association of highly-compressed, 111, 513; on the chemical combination of, 337; on the heat transmission in engine-, 883.
- Geological Society, proceedings of the, 254, 479, 619, 783.
- Geometrical note on de Sitter's world, 930.
- George (W. H.) on the Helmholtz theories of the struck string, 591.
- Germanium, on the mass-spectrum of, 394.
- Ghosh (R. N.) on the elastic impact of a pianoforte hammer, 1141.
- Gibson (Prof. A. H.) on heat dissipation from the surfaces of pipes and cylinders, 324; on the rate of heat transmission in an internal combustion engine, 883.
- Green (Dr. G.) on waves due to a single impulse in deep and shallow water, 183.
- Greenhill (Sir G.) on the steady precession of a body rolling on a table, 94.
- Greenly (Dr. E.) on the geology of the Clevedon - Portishead area, 619.
- Grindley (G. C.) on the emission from an incandescent platinum wire in air, 689.
- Gwyther (R. F.) on certain types of elastic stress, 679; on the general equations of elastic stress, 965.
- H particles from aluminium, on the, 500.
- Hafnium, on the mass-spectrum of, 398.
- Halogen hydrides, on the, 549.
- Handford (C.) on a valve method of detecting minute slipping in metals, 896.
- Havelock (Prof. T. H.) on dynamical illustrations of the pressure of radiation, 754.
- Heat dissipation from the surfaces of pipes and cylinders in an air current, on, 324.
- transmission, on the rate of, in an internal combustion engine, 883.
- Helium, on ionization by collision in, 401; on the electrical properties of, 737.
- Helmholtz theories of the struck string, on the, 591.
- Hess (Prof. V. F.) on an apparatus for purification of radium emanation, 713.
- High-vacuum pump system, on an all-metal, 918, 1016.
- Hitchins (Miss A. F. R.) on the relation between uranium and radium, 1148.
- Horton (Prof. F.) on the emission of secondary electrons from metals under electronic bombardment, 245.
- Hoyt (Dr. F. C.) on the relative probabilities of the transitions involved in the Balmer series lines of hydrogen, 826.
- Hydrides, on the halogen, 549.
- Hydrogen, on the kinetic energy of electrons from tungsten in, 458; on the spectra of the halogen compounds of, 549; on the transitions involved in the Balmer series lines of, 826.
- Hyperbolic world, geometrical note on the, 930.
- Identical relations in Einstein's theory, on the, 584.
- Inertia, on the increase of, of bodies vibrating in a liquid, 624.
- Ingram (W. H.) on the auto-transformer, 241, 1092.
- Integral $\int_0^x K_0(t)dt$, table of values of the, 736.
- Interference fringes, on a new explanation of, 453.
- Internal combustion engine, on the rate of heat transmission in an, 883.
- International Conference on Applied Mechanics, programme of the, 784.
- Iouium, on the period of, 1148.
- Ionization by electronic impacts, on statistical equilibrium and, 257; on the absence of, by electrons with speeds comparable with that of light, 306; on, by collision in helium, 401.
- Ions, on the recombination of gaseous, 337; on the emission of, from incandescent platinum, 689; on the emission of, from hot tungsten, 1025.
- Iron, on the anhyseretic qualities of, 848.

- Iron spheroid, on the charging effect produced by the rotation of a prolate, in a magnetic field, 842.
- Jacobsen (J. C.) on the rate of decay of the short-life product radium C', 23.
- Jenkin (C. F.) on damped vibrations, 303.
- Jenkins (Prof. W. A.) on the emission of positive ions from hot tungsten, 1025.
- Jevons (W.) on the origin of spectral lines hitherto attributed to oxygen, 586.
- Jones (Prof. E. T.) on valve-generated oscillations in coupled circuits, 625.
- Jones (Prof. O. T.) on the Upper Towy drainage-system, 623.
- K electrons, on the emission of, 1.
- Kaye (Dr. G. W. C.) on an all-metal high-vacuum pump system, 918, 1016.
- Kearney (Miss M. B.) on the fluorescence of æsculin solutions, 648.
- Keeley (T. C.) on a new form of electrometer, 577.
- Kirsch (Dr. G.) on the artificial disintegration of atoms, 500.
- Kramers (Dr. H. A.) on the quantum theory of radiation, 785.
- Lawrence (E.) on the charging effect produced by the rotation of a prolate iron spheroid, 842.
- Lees (E.) on an empirical equation of state for fluids, 431.
- Light quanta, on a theory of, 446, 873.
- Lindemann (A. F.) on a new form of electrometer, 577.
- Lindemann (Prof. F. A.) on a new form of electrometer, 577.
- Liquids, on the molecular association of, 111, 513; on the increase of inertia of bodies vibrating in, 624; on convective cooling in, 972, 1057.
- Lithium, on the mass-spectrum of, 393.
- Lowry (Prof. T. M.) on the electronic theory of valency, 1021.
- Lukirsky (P. I.) on soft X-rays from carbon, 466.
- Lunnon (R. G.) on the resistance of air to falling spheres, 173.
- McCallum (S. P.) on electrical properties of helium, 737.
- Mache (Prof. H.) on a diffusion theory of comets, 724.
- Magnesium, on the mass-spectrum of, 393.
- Magnetic field, on the charging effect produced by the rotation of a prolate iron spheroid in a, 842.
- Manganese, on the mass-spectrum of, 396.
- Manley (J. J.) on an automatic feeder for coloured flames, 859.
- Manometer, on a modification of Haber's quartz-fibre, 32.
- Mass-spectra of chemical elements, on the, 385.
- Mercury globules, on the effect of an electric current on the motion of, in sulphuric acid, 802.
- Metals, on the emission of secondary electrons from, 245; on a valve method for detecting slipping in, 896.
- Milne (E. A.) on statistical equilibrium in relation to the photo-electric effect, 209.
- Molecular association of liquids and highly compressed gases, on the, 111, 513.
- structure, on electric moment and, 530.
- Molecules, on the mean distance between, in a fluid, 671.
- Monomolecular reactions, on, 337.
- Mourashkinsky (B. E.) on the diffraction image of two close luminous lines of finite width, 1105.
- Neon, on the motion of electrons in, 379.
- Newman (Prof. F. H.) on the potential gradient in the sodium-potassium vapour arc lamp, 603; on the wave form of the current when an electric discharge is passed through mercury vapour, 939.
- Nickel, on the anhyseretic qualities of, 848.
- Nuclei, on the capture and loss of electrons by swift, 416.
- Oblique focal distance, on formulæ for, 864.
- Observations, on the adjustment of, 816.
- Oscillations, on valve-generated, in coupled circuits, 625.

- Osman (C. W.) on the geology of the northern border of Dartmoor, 479.
- Owen (S. P.), table of values of the integral $\int_0^x K_0(t) dt$, 736.
- Oxygen, on spectral lines hitherto attributed to, 586.
- Paraffin wax and water, on the angle of contact between, 91.
- Particle, on the relativity dynamics of a, 1158.
- Pendulum dampings, on the experimental correlation of viscosities to, 495.
- Pettersson (Dr. H.) on the artificial disintegration of atoms, 500.
- Phosphorus, on the arc spectrum of, 874.
- Photoelectric and photochemical activity, on, 975.
- effect, on statistical equilibrium in relation to the, 209.
- Pianoforte hammer, on the elastic impact of a, 1141.
- Pipes, on heat dissipation from the surfaces of, 324.
- Pitchblende, on the ionium-thorium ratio in, 1148.
- Platinum wire, on the emission from an incandescent, in air, 689.
- Polarization of resonance radiation, on the, 832.
- Potential energy, calculations of the, for some atomic models, 992.
- Precession, on the steady, of a body rolling on a table, 94.
- Pulsation of tones, on the critical frequency of, 50, 126, 141, 941.
- Pump, on an all-metal high-vacuum, 918, 1016.
- Quanta, on a theory of light, 446, 873.
- Quantum theory of radiation, on the, 785.
- Radiation, on corpuscular, excited by X-rays, 1; on some dynamical illustrations of the pressure of, 754; on the polarization of resonance, 832; on the quantum theory of, 785.
- Radium, on the relation between uranium and, 1148.
- C', on the rate of decay of, 23.
- emanation, on an apparatus for purification of, 713.
- Raman (Prof. C. V.) on the mean distance between neighbouring molecules in a fluid, 671.
- Rational units, on ultimate, 159.
- Refractivities and sizes of atoms, on the relation between, 772.
- Relativity dynamics of a particle, on the, 1158.
- Resonance radiation, on the polarization of, 832.
- Reynolds (Prof. S. H.) on the igneous rocks of the Tortworth inlier, 254; on the geology of the Clevedon-Portishead area, 619.
- Richardson (Prof. O. W.) on photoelectric and photochemical activity, 975.
- Roy (S. C.) on the emission of electrons from hot bodies, 561.
- Rubidium, on the vapour pressure of, 32.
- Russell (Dr. A. S.) on the complexity of the elements, 1121.
- Rutherford (Sir E.) on the capture and loss of electrons by alpha particles, 277.
- Saltmarsh (Miss M. O.) on the arc spectrum of phosphorus, 874.
- Scandium, on the mass-spectrum of, 396.
- Schouten (Prof. J. A.) on the identical relations in Einstein's theory, 584.
- Scott (Dr. D. H.) on the vapour pressures of caesium and rubidium, 32.
- Silberstein (Dr. L.) on the curvature invariant of space-time, 907.
- Silver, on the corpuscular radiation from, 18; on the mass-spectrum of, 398.
- Slater (J. C.) on the quantum theory of radiation, 785.
- Slipping, on a valve method for detecting, in metals, 896.
- Smyth (Dr. C. P.) on electric moment and molecular structure, 530.
- Soddy (Prof. F.) on the relation between uranium and radium, 1148.
- Sodium-potassium vapour arc lamp, on the potential gradient in the, 603.
- Solutions, on the theory of, 977.
- Space-time, on the curvature invariant of, 907.
- Spectra, on the quantum theory of, 797.
- Spectral lines, on certain, hitherto attributed to oxygen, 586.

- Spectrum, on the arc, of phosphorus, 874.
- Statistical equilibrium, on, in relation to the photoelectric effect, 209; on, and ionization by electronic impacts, 257.
- String, on the vibration of a, 754; on the Helmholtz theories of the struck, 591.
- Strontium, on the mass-spectrum of, 395.
- Struik (Dr. D. J.) on the identical relations in Einstein's theory, 584.
- Sulphuric acid, on the effect of an electric current on the motion of mercury globules in, 802.
- Swann (Prof. W. F. G.) on the absence of ionization by electrons with speeds comparable with that of light, 306.
- Synge (J. L.) on the influence of the earth's rotation on a top, 525.
- Thermionic currents, on, 561.
- emission, on the thermodynamics of, 544.
- Thermodynamics, on a supposed limitation of the second law of, 152, 779.
- Thomas (Dr. J. S. G.) on the entrainment of air by a jet of gas, 1048.
- Thomas (W. N.) on damped vibrations, 303.
- Thomson (Sir J. J.) on the recombination of gaseous ions, the chemical combination of gases, and monomolecular reactions, 337.
- Timoschenko (Prof. S.) on two-dimensional problems in elasticity, 1095.
- Tin, on the corpuscular radiation from, 21.
- Titanium, on the mass-spectrum of, 397.
- Tones, on the critical frequency of pulsation of, 50, 126, 141, 941.
- Top, on the influence of the earth's rotation on a, 525.
- Townsend (Prof. J. S.) on ionization by collision in helium, 401; on the electrical properties of helium, 737.
- Tubes, on bending stresses in thin-walled, 197; on the secondary failure of thin, 319.
- Tungsten, on the emission of positive ions from hot, 1025.
- Tungsten filament, on the kinetic energy of electrons emitted from a, 458.
- Tyndall (Prof. A. M.) on the emission from an incandescent platinum wire in air, 689.
- Ultra-violet, on lines in the, hitherto attributed to oxygen, 586.
- Units, on ultimate rational, 159.
- Uranium, on the relation between radium and, 1148.
- Valency, on the electronic theory of, 569, 1021.
- bond, on the, 702.
- Valve circuits, on the application of oscillating, to the measurement of physical quantities, 66.
- generated oscillations in coupled circuits, on, 625.
- method of detecting slipping in metals, on a, 896.
- Vanadium, on the mass-spectrum of, 397.
- Vibration frequency of an element, on the characteristic, 84.
- galvanometer, on the anomalous behaviour of a, 609.
- Vibrations, on damped, 303.
- Viscosities, on the experimental correlation of to pendulum dampings, 495.
- Wagstaff (J. E. P.) on the application of oscillating valve circuits to the measurement of physical quantities, 66; on the characteristic vibration frequency of an element, 84; on the effect of an electric current on the motion of mercury globules in dilute sulphuric acid, 802.
- Walker (Dr. E. E.) on the molecular association of liquids and highly-compressed gases, 111, 513.
- Wallis (Dr. F. S.) on the Avonian of the Tytherington-Tortworth-Wickwar ridge, 620.
- Water, on the angle of contact between paraffin wax and, 91.
- Waves due to a single impulse, on, 183.
- Weinberg (Miss M.) on the critical frequency of pulsation of tones, 50, 126; on the effect of fatiguing the ear with combinations of two or more tones, 141.
- Wills (Dr. L. J.) on the development of the Severn valley, 254.

- Witmer (Prof. E. E.) on a supposed limitation of the second law of thermodynamics, 152.
- Woodward (Miss I.) calculations of the potential energy for some atomic models, 992.
- Wright (W. B.) on the Lough Neagh clays, 783.
- Wrinch (Dr. D.), tables of Bessel functions, 62.
- Wrinch (H. E. H.), tables of Bessel functions, 62.
- X-rays, on corpuscular radiation excited by, 1; on soft, from carbon, 466.
- Yttrium, on the mass spectrum of, 398.
- Zero sum, on the method of, 816.
- Zirconium, on the mass-spectrum of, 398.

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